Delta-shocks in the Navier-Stockes system of granular hydrodynamics

Vladimir Shelkovich

St.-Petersburg State Architecture and Civil Engineering University, St. Petersburg, Russia

The points to be considered are integral identities for the definition of multidimensional δ-shocks to the system of granular hydrodynamics, the Rankine–Hugoniot conditions for δ-shocks, the transportation and concentration processes related to δ-shocks, and propagation of a δ-shock wave. To deal with these singular problems the *weak asymptotics method* developed in [1]– [3], [14] is used.

δ-SHOCK TYPE SOLUTIONS

Definition

Consider the following hydrodynamics system of granular gas

$$
\begin{aligned}
\rho_t + \nabla\cdot(\rho U) &= 0,\\
(\rho U)_t + \nabla\cdot(\rho U\otimes U + I\,\rho T) &= 0,\\
T_t + \nabla\cdot(UT) &= -(\gamma-2)T\,\nabla\cdot U - \Lambda\rho T^{3/2},
\end{aligned}
\tag{1}
$$

where ρ is the gas density, U is the velocity, T is the temperature, $p=\rho T$ is the pressure; γ is the adiabatic index (if $n=2$ then $\gamma=2$, and if $n=3$, then $\gamma=5/3$), Λ is a constant connected with the energy of collision processes (which can be calculated in the framework of the kinetic theory). As will be shown (on the physical level of rigor) in [4], [5] that system (1) can admit a solution which contains δ-function in the density.

- $\Gamma=\{(x,t): S(x,t)=0\}$ is a hypersurface in the upper half-space $\{(x,t): x\in\mathbb{R}^n,\ t\in[0,\infty)\}\in\mathbb{R}^{n+1}$. $\Gamma_t=\{x: S(x,t)=0\}$ is a moving hypersurface in $\mathbb{R}^n$. $\nu=\frac{\nabla S}{|\nabla S|}$ is the unit space normal to the surface Γ_t pointing from $\Omega_t^-=\{x\in\mathbb{R}^n: S(x,t)<0\}$ to $\Omega_t^+=\{x\in\mathbb{R}^n: S(x,t)>0\}$. $-G=\frac{S_t}{|\nabla S|}$ is the velocity (along the normal ν) of the moving wave front Γ_t.
- Consider *δ-shock type initial data*

$$
\begin{gathered}
\big(U^0(x),\rho^0(x),T^0(x),\ x\in\mathbb{R}^n;\ U_\delta^0(x),\ x\in\Gamma_0\big),\\
\text{where } \rho^0(x)=\widehat{\rho}^0(x)+e^0(x)\delta(\Gamma_0),
\end{gathered}
\tag{2}
$$

and $U^0\in L^\infty(\mathbb{R}^n;\mathbb{R}^n)$, $\widehat{\rho}^0, T^0\in L^\infty(\mathbb{R}^n;\mathbb{R})$, $e^0\in C(\Gamma_0)$, $\Gamma_0=\{x: S^0(x)=0\}$ is the initial position of the δ-shock wave front, $\nabla S^0(x)\big|_{S^0=0}\neq 0$, $U_\delta^0(x)$, $x\in\Gamma_0$ is the *initial velocity* of the δ-shock; $\delta(\Gamma_0)$ ($\equiv\delta(S^0)$) is the Dirac delta function concentrated on the Γ_0.

Definition 1. Distributions (U,ρ,T) and a hypersurface Γ, where

$$\rho(x,t)=\widehat{\rho}(x,t)+e(x,t)\delta(\Gamma),$$

$U\in L^\infty(\mathbb{R}^n\times(0,\infty);\mathbb{R}^n)$, $\widehat{\rho}, T\in L^\infty(\mathbb{R}^n\times(0,\infty);\mathbb{R})$, $e\in C(\Gamma)$, is called a *δ-shock wave type solution* of the Cauchy problem (1), (2) if for all $\varphi\in\mathcal{D}(\mathbb{R}^n\times[0,\infty))$

$$
\begin{aligned}
&\int_0^\infty\!\!\int \widehat{\rho}\Big(\varphi_t+U\cdot\nabla\varphi\Big)\,dx\,dt+\int_\Gamma e\frac{\delta\varphi}{\delta t}\frac{d\Gamma}{\sqrt{1+G^2}}\\
&\qquad+\int\widehat{\rho}^0(x)\varphi(x,0)\,dx+\int_{\Gamma_0}e^0(x)\varphi(x,0)\,d\Gamma_0=0,\\
&\int_0^\infty\!\!\int\Big(\widehat{\rho}U\Big(\varphi_t+U\cdot\nabla\varphi\Big)+\widehat{\rho}T\nabla\varphi\Big)\,dx\,dt+\int_\Gamma eU_\delta\frac{\delta\varphi}{\delta t}\frac{d\Gamma}{\sqrt{1+G^2}}\\
&\qquad+\int U^0(x)\widehat{\rho}^0(x)\varphi(x,0)\,dx+\int_{\Gamma_0}e^0(x)U_\delta^0(x)\varphi(x,0)\,d\Gamma_0=0,\\
&\int_0^\infty\!\!\int T\Big(\varphi_t+U\cdot\nabla\varphi\Big)\,dx\,dt+\int T^0(x)\varphi(x,0)\,dx\\
&\qquad-\int_{\mathbb{R}_+^{n+1}\setminus\Gamma}\Big((\gamma-2)T\,\nabla\cdot U+\Lambda\rho T^{3/2}\Big)\varphi\,dx\,dt\\
&=-\int_\Gamma\Big((2-\gamma)[U]\cdot\nu\big(T^++a[T]\big)+\Lambda e\big((T^+)^{3/2}+b[T^{3/2}]\big)\Big)\frac{\varphi\,d\Gamma}{\sqrt{1+G^2}},
\end{aligned}
$$

where $a,b\in(0,\,1)$ are constants which are defined from the initial data and physical considerations; $U_\delta=\nu G=-\frac{S_t\nabla S}{|\nabla S|^2}$ is the the velocity of the δ-shock wave front Γ_t; $\frac{\delta\varphi}{\delta t}\stackrel{def}{=}\frac{\partial\varphi}{\partial t}+G\frac{\partial\varphi}{\partial\nu}$ $\Big(=\frac{\partial\varphi}{\partial t}+U_\delta\cdot\nabla\varphi=\frac{D\varphi}{Dt}\Big)$ is the δ-derivative of φ with respect to the time variable t [6, 5.2.,12.], $\frac{\partial\varphi}{\partial\nu}=\nu\cdot\nabla\varphi$ is the normal derivative.

The Rankine–Hugoniot conditions

Theorem 1. *Let $\Omega\subset\mathbb{R}^n\times(0,\infty)$ be a region cut by a hypersurface $\Gamma=\{(x,t): S(x,t)=0\}$ into left- and right-hand parts $\Omega^\mp=\{(x,t):\mp S(x,t)>0\}$. Let (U,ρ,T), Γ be a δ-shock solution of* (1). *Suppose that U,ρ,T are smooth in $\Omega^\pm$ and have one-sided limits $U^\pm$, $\widehat{\rho}^\pm$, $T^\pm$ on Γ. Then the* Rankine–Hugoniot conditions *hold:*

$$
\begin{aligned}
\frac{\delta e}{\delta t}+\nabla_{\Gamma_t}\cdot(eU_\delta) &= \big([\rho U]-[\rho]U_\delta\big)\cdot\nu,\\
\frac{\delta(eU_\delta)}{\delta t}+\nabla_{\Gamma_t}\cdot(eU_\delta\otimes U_\delta) &= \big([\rho U\otimes U+I\,\rho T]-[\rho U]U_\delta\big)\cdot\nu,\\
\big([TU]-[T]U_\delta\big)\cdot\nu &= (\gamma-2)[U]\cdot\nu\big(T^++a[T]\big)+\\
&\quad+\Lambda e\big((T^+)^{3/2}+b[T^{3/2}]\big),
\end{aligned}
$$

where $\nabla_{\Gamma_t}=\Big(\frac{\delta}{\delta x_1},\dots,\frac{\delta}{\delta x_n}\Big)$, $\frac{\delta f}{\delta x_j}\stackrel{def}{=}\big(\frac{\partial f}{\partial x_j}-\nu_j\frac{\partial f}{\partial\nu}\big)\big|_\Gamma$ is the δ-derivative of f with respect to x_j, $j=1,\dots,n$ ([6, 5.2.,12.]).

Here $\nabla_{\Gamma_t}\cdot(eU_\delta)=-2\mathcal{K}Ge$, $\nabla_{\Gamma_t}\cdot(eU_\delta\otimes U_\delta)=-2\mathcal{K}GeU_\delta$, where $\mathcal{K}\stackrel{def}{=}-\frac{1}{2}\nabla_{\Gamma_t}\cdot\nu=-\frac{1}{2}\nabla\cdot\nu$ is the mean curvature of the front Γ_t.

- In the direction ν the characteristic equation of (1) has repeated eigenvalues $\lambda=U\cdot\nu$. Therefore we assume that for a solution of the Cauchy problem (1), (2) the *geometric entropy condition* holds:

$$U^+(x,t)\cdot\nu\big|_\Gamma<U_\delta(x,t)\cdot\nu\big|_\Gamma<U^-(x,t)\cdot\nu\big|_\Gamma,\tag{3}$$

where U_δ is the velocity of the δ-shock, $U^\pm$ are the velocities behind the δ-shock front and ahead of it; ν is the unit space normal of Γ.

TRANSPORTATION AND CONCENTRATION PROCESSES

Let (U,ρ,T) and Γ be a solution of the Cauchy problem (1), (2) compactly supported with respect to x, and $\Omega_t^\pm=\{x\in\mathbb{R}^n: \pm S(x,t)<0\}$. Let

$$M(t)=\int_{\Omega_t^-\cup\Omega_t^+}\rho(x,t)\,dx,\quad P(t)=\int_{\Omega_t^-\cup\Omega_t^+}\rho(x,t)U(x,t)\,dx;$$

$$m(t)=\int_{\Gamma_t}e(x,t)\,d\Gamma_t,\quad p(t)=\int_{\Gamma_t}e(x,t)U_\delta(x,t)\,d\Gamma_t;$$

be masses, momentums of the domain $\Omega_t^-\cup\Omega_t^+$, and of front Γ_t.

Theorem 2. *Let (U,ρ,T) and $\Gamma=\{(x,t): S(x,t)=0\}$ be a δ-shock wave type solution of the Cauchy problem* (1), (2), (3), *compactly supported with respect to x, where $\rho(x,t)=\widehat{\rho}(x,t)+e(x,t)\delta(\Gamma)$. Suppose that (U,ρ,T) is smooth in $\Omega_\pm$. Then transport processes between the region outside of the δ-shock front $\Omega_t^-\cup\Omega_t^+=\{x\in\mathbb{R}^n: S(x,t)\neq0\}$ and the δ-shock front Γ_t are going on in such a way that*
(a) the total mass $M(t)+m(t)$ and momentum $P(t)+p(t)$ are independent of time,
(b) the mass concentration process on the moving wave front Γ_t takes place: $\dot{m}(t)\geq0$,

PROPAGATION OF δ-SHOCKS

Consider system (1) with the initial data $(U^0,\rho^0,T^0,U_\delta^0)$:

$$
\begin{aligned}
U^0(x) &= U^{0+}(x)+[U^0(x)]\Theta(-\Gamma_0),\\
\rho^0(x) &= \rho^{0+}(x)+[\rho^0(x)]\Theta(-\Gamma_0)+e^0(x)\delta(\Gamma_0),\\
T^0(x) &= T^{0+}(x)+[T^0(x)]\Theta(-\Gamma_0),
\end{aligned}
\tag{4}
$$

$\Gamma_0=\{x: S^0(x)=0\}$ is the initial position of the δ-shock front. Θ is the Heaviside function. We assume that (3) holds for $t=0$.
We seek a δ-shock wave type solution of this Cauchy problem in the form

$$
\begin{aligned}
U(x,t) &= U^+(x,t)+[U(x,t)]\Theta(-\Gamma),\\
\rho(x,t) &= \rho^+(x,t)+[\rho(x,t)]\Theta(-\Gamma)+e(x,t)\delta(\Gamma),\\
T(x,t) &= T^+(x,t)+[T(x,t)]\Theta(-\Gamma),
\end{aligned}
\tag{5}
$$

where vector-functions $U^\pm$ and functions $\rho^\pm, T^\pm\geq0$, $e\geq0$, S are to be found, $U^-=U^++[U]$, $\rho^-=\rho^++[\rho]$, $T^-=H^++[H]$.

- At first, we construct a weak asymptotic solution of the Cauchy problem (1), (4), which is a triple $(U_\varepsilon(x,t),\rho_\varepsilon(x,t),T_\varepsilon(x,t))$ such that U_ε, ρ_ε, T_ε are smooth as $\varepsilon>0$, $x\in\mathbb{R}^n$, $t\in[0,T]$, and satisfies the system

$$
\begin{aligned}
(\rho_\varepsilon)_t+\nabla\cdot(\rho_\varepsilon U_\varepsilon) &= o_{\mathcal{D}'}(1),\\
(\rho_\varepsilon U_\varepsilon)_t+\nabla\cdot(\rho_\varepsilon U_\varepsilon\otimes U_\varepsilon+I\,\rho_\varepsilon T_\varepsilon) &= o_{\mathcal{D}'}(1),\\
(T_\varepsilon)_t+\nabla\cdot(T_\varepsilon U_\varepsilon)+(\gamma-2)T_\varepsilon\,\nabla\cdot U_\varepsilon+\Lambda\rho_\varepsilon T_\varepsilon^{3/2} &= o_{\mathcal{D}'}(1),\\
U_\varepsilon(x,0)=U^0(x)+o_{\mathcal{D}'}(1),&\\
\rho_\varepsilon(x,0)=\rho^0(x)+o_{\mathcal{D}'}(1),&\\
T_\varepsilon(x,0)=T^0(x)+o_{\mathcal{D}'}(1),&\ \ \varepsilon\to+0,
\end{aligned}
\tag{6}
$$

$O_{\mathcal{D}'}(\varepsilon^\alpha)$, $\varepsilon\to+0$, is a collection of distributions $f_\varepsilon(\cdot,t)\in\mathcal{D}'(\mathbb{R}^n)$, $t\in[0,T]$, $\varepsilon>0$ such that $\langle f_\varepsilon(\cdot,t),\psi(\cdot)\rangle=O(\varepsilon^\alpha)$, for any $\psi\in\mathcal{D}(\mathbb{R}^n)$, where $\langle f_\varepsilon(\cdot,t),\psi(\cdot)\rangle$ is continuous in t, $O(\varepsilon^\alpha)$ is *uniform* in $t\in[0,T]$.

- A *weak asymptotic solution* is constructed as a smooth ansatz

$$
\begin{aligned}
u_{j\varepsilon}(x,t) &= \widetilde{u}_{j\varepsilon}(x,t)+R_j(x,t,\varepsilon), \quad j=1,\dots,n,\\
\rho_\varepsilon(x,t) &= \widetilde{\rho}_\varepsilon(x,t)+R_\rho(x,t,\varepsilon), \qquad \varepsilon>0.\\
T_\varepsilon(x,t) &= \widetilde{H}_\varepsilon(x,t)+R_H(x,t,\varepsilon),
\end{aligned}
\tag{7}
$$

$(\widetilde{U}_\varepsilon,\widetilde{\rho}_\varepsilon,\widetilde{T}_\varepsilon)$ is the regularization of singular ansatz (5) with respect to singularities $\Theta(-S)$, $\delta(S)$; and the corrections $R_\alpha(x,t,\varepsilon)$, are desired functions admitting estimates: $R_\alpha(x,t,\varepsilon)=o_{\mathcal{D}'}(1)$, $\alpha=1,\dots,n;\rho;T$. We set

$$
\begin{aligned}
\widetilde{u}_{j\varepsilon}(x,t) &= u_j^++[u_j]\Theta_j\big(-S,\varepsilon\big), \quad j=1,\dots,n,\\
\widetilde{\rho}_\varepsilon(x,t) &= \rho^++[\rho]\Theta_\rho\big(-S,\varepsilon\big)+e\delta_\rho\big(S,\varepsilon\big),\\
\widetilde{T}_\varepsilon(x,t) &= T^++[T]\Theta_T\big(-S,\varepsilon\big),
\end{aligned}
\tag{8}
$$

where $\delta_\rho(S,\varepsilon)$ is a regularization of the delta function $\delta(S)$ and $\Theta_j\big(S,\varepsilon\big)$, $\Theta_\rho\big(S,\varepsilon\big)$, $\Theta_T\big(S,\varepsilon\big)$ are regularizations of the Heaviside function $\Theta(S)$.

- Substituting the smooth ansatz (7), (8) into system (1), we obtain the necessary and sufficient conditions for $U^\pm$, $\rho^\pm$, $T^\pm$, e, S, and corrections $R_\alpha(x,t,\varepsilon)$ to satisfy system (6).
- We find a δ-shock solution of the Cauchy problem (1), (4) as the weak limit

$$U=\lim_{\varepsilon\to+0}U_\varepsilon,\quad \rho=\lim_{\varepsilon\to+0}\rho_\varepsilon,\quad T=\lim_{\varepsilon\to+0}T_\varepsilon.$$

- Next, we prove that (U,ρ,T) satisfies the identities of Definition 1.

Theorem 3. *Let the entropy condition* (3) *hold. Then there exist $T>0$ and a zero neighborhood $K\subset\mathbb{R}^n$ such that for $(x,t)\in K\times[0,\,T)$, the Cauchy problem* (1), (4) *has a unique solution*

$$
\begin{aligned}
U(x,t) &= U^+(x,t)+[U(x,t)]\Theta(-\Gamma),\\
\rho(x,t) &= \rho^+(x,t)+[\rho(x,t)]\Theta(-\Gamma)+e(x,t)\delta(\Gamma),\\
T(x,t) &= H^+(x,t)+[T(x,t)]\Theta(-\Gamma),
\end{aligned}
$$

which satisfies the integral identities of Definition 1, *where $U^\pm$, $\rho^\pm$, $T^\pm$, e, U_δ satisfy the system of equations*

$$
\begin{aligned}
(\rho^\pm)_t+\nabla\cdot(\rho^\pm U^\pm)=0,&\quad (x,t)\in\Omega^\pm,\\
(\rho^\pm U^\pm)_t+\nabla\cdot(\rho^\pm U^\pm\otimes U^\pm+I\,\rho^\pm T^\pm)=0,&\quad (x,t)\in\Omega^\pm,\\
(T^\pm)_t+\nabla\cdot(T^\pm U^\pm)+(\gamma-2)T^\pm\,\nabla\cdot U^\pm+\Lambda\rho^\pm(T^\pm)^{3/2}=0,&\quad (x,t)\in\Omega^\pm,\\
\tfrac{\delta e}{\delta t}+\nabla_{\Gamma_t}\cdot(eU_\delta)=\big([\rho U]-[\rho]U_\delta\big)\cdot\nu,&\quad (x,t)\in\Gamma\\
\tfrac{\delta(eU_\delta)}{\delta t}+\nabla_{\Gamma_t}\cdot(eU_\delta\otimes U_\delta)=\big([\rho U\otimes U+I\,\rho T]-[\rho U]U_\delta\big)\cdot\nu,&\quad (x,t)\in\Gamma\\
\big([TU]-[T]U_\delta\big)\cdot\nu=(\gamma-2)[U]\cdot\nu\big(T^++a[T]\big)+\Lambda e\big((T^+)^{3/2}+b[T^{3/2}]\big),&\quad (x,t)\in\Gamma
\end{aligned}
$$

with the initial data defined from (4), $S(x,0)=S^0(x)$.

THE PHYSICAL CONTEXT OF δ-SHOCKS

Systems of conservation laws admitting δ-shocks have a physical context and are used in applications.

- Zero-pressure gas dynamics in the form

$$\rho_t+\nabla\cdot(\rho U)=0,\qquad (\rho U)_t+\nabla\cdot(\rho U\otimes U)=0$$

is used to describe the formation of large-scale structures of the universe [13], [15].

- Zero-pressure gas dynamics in the form

$$
\begin{aligned}
\rho_t+\nabla\cdot(\rho U) &= 0,\\
(\rho U)_t+\nabla\cdot(\rho U\otimes U) &= 0,\\
\Big(\tfrac{\rho|U|^2}{2}+H\Big)_t+\nabla\cdot\Big(\Big(\tfrac{\rho|U|^2}{2}+H\Big)U\Big) &= 0,
\end{aligned}
$$

($H(x,t)$ is the internal energy) is used for modeling of dusty gases [7], [8]. The problems related with δ-shocks in the above system were considered in [10], [11].

- δ-Shocks arise in the model of non-classical shallow water flows [12]:

$$\eta_t+(\eta u)_x=0,\qquad u_t+uu_x+\frac{1}{F^2}\eta_x=0$$

with large Froude number $F>>1$, where η is depth, u is horizontal velocity. This system is a model of the impact of two fluids layers in the situation where the flow can be modeled as two smooth regions joined by the singularity in the flow field.

- δ-Shocks arise in nonlinear chromatography [9]:

$$\Big(u_j+\frac{a_ju_j}{1-u_1+u_2}\Big)_t+(u_j)_x=0,\quad u_j\geq0,\quad x\geq0,\ t\geq0,\quad j=1,2.$$

References

[1] V.G. Danilov, G.A. Omel′yanov, V.M. Shelkovich, *Weak Asymptotics Method and Interaction of Nonlinear Waves*, in Mikhail Karasev (ed.), “Asymptotic Methods for Wave and Quantum Problems”, Amer. Math. Soc. Transl. Ser. 2, **208**, 2003, 33-165.

[2] V.G. Danilov, V.M. Shelkovich, *Delta-shock wave type solution of hyperbolic systems of conservation laws*, Quart. Appl. Math., **63**, no. 3, (2005), 401-427.

[3] V.G. Danilov, V.M. Shelkovich, *Dynamics of propagation and interaction of delta-shock waves in conservation law systems*, Journal of Differential Equations, **211**, (2005), 333-381.

[4] I. Fouxon, B. Meerson, M. Assaf, and E. Livne, *Formation of density singularities in ideal hydrodynamics of freely cooling inelastic gases: A family of exact solutions*, Phys. Fluids, **19**, 093303 (2007), (17 pages).

[5] I. Fouxon, B. Meerson, M. Assaf, and E. Livne, *Formation of density singularities in hydrodynamics of inelastic gases*, Phys. Review, E **75**, 050301(R) (2007), (4 pages).

[6] Ram P. Kanwal, *Generalized Functions: Theory and technique*, Birkhäuser Boston–Basel–Berlin, 1998.

[7] A.N. Kraiko, *Discontinuity surfaces in medium without self-pressure*, Prikladnaia Matematika i Mekhanika, **43**, (1979), 539–449. (In Russian)

[8] A.N. Kraiko, *On two-phase flows model of gas and dispersed in it particles*, Prikladnaia Matematika i Mekhanika, **46**, issue 1, (1982), 96–106. (In Russian)

[9] M. Mazzotti, *Nonclassical composition fronts in nonlinear chromatography: delta-shock*, Ind. Eng. Chem. Res., **48**, (2009), 7733-7752.

[10] B. Nilsson, V. M. Shelkovich, *Mass, momentum and energy conservation laws in zero-pressure gas dynamics and δ-shocks*, Applicable Analysis, Vol. 90, No. 11, (2011), 1677–1689.

[11] B. Nilsson, O. S. Rozanova, V. M. Shelkovich, *Mass, momentum and energy conservation laws in zero-pressure gas dynamics and δ-shocks. II*, Applicable Analysis, Vol. 90, No. 5, (2011), 831–842.

[12] C. M. Edwards, S. D. Howinson, H. Ockendon and J. R. Ockendon, *Non-classical shallow water flows*, Journal of Applied Mathematics, **73**, (2008), 137–157.

[13] S.F. Shandarin and Ya.B. Zeldovich, *The large-scale structure of the universe: turbulence, intermittence, strucrures in self-gravitating medium*, Rev. Mod. Phys., **61**, (1989), 185–220.

[14] V.M. Shelkovich, *δ- and δ′-shock types of singular solutions to systems of conservation laws and the transport and concentration processes*, Uspekhi Mat. Nauk, **63**:3(381), (2008), 73–146. English transl. in Russian Math. Surveys, **63**:3, (2008), 473–546.

[15] Ya.B. Zeldovich, *Gravitationnal instability: An approximate theory for large density perturbations*, Astron. Astrophys., **5**, (1970), 84–89.