Delta-shocks in the Navier-Stockes system of granular hydrodynamics

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1. Strong singular solutions and physical models. It is well known that there are "nonclassical" situations where the Cauchy problem for a system of conservation laws admits δ-shocks, which are solutions whose components contain Dirac delta functions. In contrast to the classical shock wave discontinuities, δ-shocks *carry mass, momentum and energy* and are related with *transport and concentration processes.* In numerous papers, δ-shocks were studied in the zero-pressure gas dynamics. This system was used to describe the formation of large-scale structures of the universe, for modeling "dusty" media and double-fluid mixtures of gas and solid particles. Systems of conservation laws admitting δ-shocks were used for modeling the formation and evolution of traffic jams, in nonlinear chromatography, in the model of non-classical shallow water flows.

2. δ-Shocks in granular hydrodynamics. Nowadays problems related with *granular gases* are very attractive for experimental, numerical, and theoretical investigation (see [1], [2] and the references therein). So far there is no consensus on the description of these type of media. In contrast to ordinary gases, granular gases are dilute assemblies of hard spheres which lose energy at collisions. In such gases a *local density can significantly increase while a local pressure can fall drastically.* A description of these phenomena is provided by the Navier-Stockes granular hydrodynamics which is derivable, under certain assumptions, from the basic theory. In [5], [6] (see also [2; p.60-75]), the following hydrodynamics system of granular gas

$$\begin{array}{rcl} \rho_t + \nabla\cdot(\rho U) &=& 0, \\ (\rho U)_t + \nabla\cdot(\rho U\otimes U + I\,\rho T) &=& 0, \\ T_t + \nabla\cdot(UT) + (\gamma-2)T\,\nabla\cdot U &=& -\Lambda\rho T^{3/2}, \end{array} \qquad (1)$$

was studied, where I is the identity matrix, $\otimes$ is the tensor product of vectors, ρ is gas density, U is velocity, T is temperature, $p=\rho T$ is pressure; γ is the adiabatic index (if $n=2$ then $\gamma=2$, and if $n=3$, then $\gamma=5/3$), Λ is a constant connected with the energy of collision processes. As was proved in [5], [7], solutions of system (1) generically lose the initial smoothness within a finite time. Moreover (see [5], [6]), system (1) can admit a solution which contains δ-function in the density ρ: $\rho(x,t) = 2m_*(t)\delta(x) + \rho_*(x,t)$, and $m_*(t)$, $\rho_*(x,t)$ are smooth.

Here we shall consider some problems connected with δ-shocks in system (1). To deal with δ-shocks, we will use the *weak asymptotics method* developed in [3], [4] (see also [8]).

Let $\Gamma = \big\{(x,t) : S(x,t)=0\big\}$ be a hypersurface of codimension 1 in $\big\{(x,t) : x\in\mathbb{R}^n, t\in[0,\infty)\big\} \subset \mathbb{R}^{n+1}$, $S\in C^\infty(\mathbb{R}^n\times[0,\infty))$, with $\nabla S(x,t)|_{S=0} \neq 0$

for any fixed t. Let $\Gamma_t = \{x \in \mathbb{R}^n : S(x,t) = 0\}$ be a moving surface in $\mathbb{R}^n$. Denote by $\nu = \frac{\nabla S}{|\nabla S|}$ the unit space normal to the surface Γ_t pointing from $\Omega_t^- = \{x \in \mathbb{R}^n : S(x,t) < 0\}$ to $\Omega_t^+ = \{x \in \mathbb{R}^n : S(x,t) > 0\}$. The time component of the normal vector $-G = \frac{S_t}{|\nabla S|}$ is the *velocity of the wave front* Γ_t along the space normal ν. For system (1) we consider the *δ-shock type initial data*

$$\begin{array}{l} \big(U^0(x), \rho^0(x), T^0(x),\, x \in \mathbb{R}^n;\, U_\delta^0(x),\, x \in \Gamma_0\big), \\ \qquad\qquad\qquad\qquad \text{where} \quad \rho^0(x) \;=\; \widehat{\rho}^0(x) + e^0(x)\delta(\Gamma_0), \end{array} \tag{2}$$

and $U^0 \in L^\infty(\mathbb{R}^n;\mathbb{R}^n)$, $\widehat{\rho}^0, T^0 \in L^\infty(\mathbb{R}^n;\mathbb{R})$, $e^0 \in C(\Gamma_0)$, $\Gamma_0 = \{x : S^0(x) = 0\}$ is the initial position of the δ-shock wave front, $U_\delta^0(x)$ is the *initial velocity* of the δ-shock, $\delta(\Gamma_0)$ $(\equiv \delta(S^0))$ is the Dirac delta function on Γ_0.

3. Rankine–Hugoniot conditions. First, basing on [8] we introduce the integral identities, which give a *definition of δ-shock wave type solution* of the Cauchy problem (1), (2). This solution is a triple of distributions (U, ρ, T) and a hypersurface Γ, where $\rho(x,t)$ is represented as a sum

$$\rho(x,t) = \widehat{\rho}(x,t) + e(x,t)\delta(\Gamma),$$

$U \in L^\infty(\mathbb{R}^n \times (0,\infty);\mathbb{R}^n)$, $\widehat{\rho}, T \in L^\infty(\mathbb{R}^n \times (0,\infty);\mathbb{R})$, $e \in C(\Gamma)$, and $\delta(\Gamma)$ $(\equiv \delta(S))$ is the Dirac delta function concentrated on the surface Γ. Next, using the above integral identities and repeating the proof of [8; Theorem 9.1] almost word for word, we derive the corresponding Rankine–Hugoniot conditions.

4. Mass, momentum, and energy transport laws. Assume that a moving δ-shock wave front $\Gamma_t = \{x : S(x,t) = 0\}$ permanently separates $\mathbb{R}_x^n$ into two parts $\Omega_t^\pm = \{x \in \mathbb{R}^n : \pm S(x,t) > 0\}$. Let (U, ρ, T) be compactly supported with respect to x. Denote by $M(t) = \int_{\Omega_t^- \cup \Omega_t^+} \rho(x,t)\,dx$, $m(t) = \int_{\Gamma_t} e(x,t)\,d\Gamma_t$, and $P(t) = \int_{\Omega_t^- \cup \Omega_t^+} \rho(x,t)U(x,t)\,dx$, $p(t) = \int_{\Gamma_t} e(x,t)U_\delta(x,t)\,d\Gamma_t$, masses and momenta of the region $\Omega_t^- \cup \Omega_t^+$ and the moving δ-shock wave front Γ_t, respectively, where e is a density of the wave front Γ_t, $U_\delta = \nu G = -\frac{S_t \nabla S}{|\nabla S|^2}$ is the δ-shock wave velocity. Let $W_{kin}(t) = \int_{\Omega_t^- \cup \Omega_t^+} \rho(x,t)|U(x,t)|^2/2\,dx$, $w_{kin}(t) = \int_{\Gamma_t} e(x,t)|U_\delta(x,t)|^2/2\,d\Gamma_t$, be the kinetic energies of the region $\Omega_t^- \cup \Omega_t^+$ and the moving wave front Γ_t, respectively.

Using technique of the papers [9], [8], we prove the theorem with gives *the mass, momentum and energy balance relations between the area outside of the moving δ-shock wave front and this front*, i.e., we derive connections between quantities $M(t)$ and $m(t)$, $P(t)$ and $p(t)$, $W_{kin}(t)$ and $w_{kin}(t)$.

5. Propagation of a δ-shock wave. Let S^0 be a given smooth function. Denote by $\Omega_0^\pm = \{x \in \mathbb{R}^n : \pm S^0(x) > 0\}$ the domains on the one side and on the other side of the hypersurface $\Gamma_0 = \{x \in \mathbb{R}^n : S^0(x) = 0\}$. In order to study the propagation of a singular front Γ_t starting from the initial position Γ_0, we need to solve the Cauchy problem for system (1) with the following initial data

$$\begin{array}{rcl} (U^0, \rho^0, T^0, U_\delta^0),\ \text{where}\ U^0 &=& U^{0+} + [U^0]H(-\Gamma_0), \\ \rho^0 &=& \rho^{0+} + [\rho^0]H(-\Gamma_0) + e^0(x)\delta(\Gamma_0), \\ T^0 &=& T^{0+} + [T^0]H(-\Gamma_0), \end{array} \tag{3}$$

where $U^{0-}(x) = U^{0+}(x) + [U^0(x)]$, $\rho^{0-}(x) = \rho^{0+}(x) + [\rho^0(x)]$, $T^{0-}(x) = T^{0+}(x) + [T^0(x)]$; e^0, $\rho^{0\pm}$, $T^{0\pm}$ are given functions, $U^{0\pm}$ are given vectors; $H(-\Gamma_0)$ ($\equiv H(-S^0)$) is the Heaviside function. Since in the direction ν the characteristic equation of system (1) has repeated eigenvalues $\lambda = U \cdot \nu$, we assume that for the initial data (2) the *geometric entropy condition* holds: $U^{0+}(x) \cdot \nu^0|_{\Gamma_0} < U^0_\delta(x) \cdot \nu^0|_{\Gamma_0} < U^{0-}(x) \cdot \nu^0|_{\Gamma_0}$, where $\nu^0 = \frac{\nabla S^0(x)}{|\nabla S^0(x)|}$ is the unit normal of Γ_0, U^0_δ is the *initial velocity* of the δ-shock.

Using the *weak asymptotics method* we describe the propagation of δ-shock wave, i.e., we construct a solution of the Cauchy problem (1), (3).

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