Analytical Solution of Second-Order Hyperbolic Telegraph Equations by Homotopy Analysis Method

Amit Tomar

Indian Institute of Technology, Roorkee, India

Joint work with: Dr.Rajan Arora and Prof.V.P.Singh

ABSTRACT

The homotopy analysis method is applied to obtain the solutions of the initial value problem of hyperbolic type which is called telegraph equation. This analytic technique is valid for dealing with the nonlinearity and provides a convenient way of controlling the convergence region and rate of the series solution. The results obtained by the present method are compared with exact solutions. The results reveal that the implemented technique is very effective and convenient for solving nonlinear partial differential equations. Some illustrative examples are presented to show the efficiency of the method.

HOMOTOPY ANALYSIS METHOD

Let us consider the differential equation

$$N[u(x,t)] = 0, \qquad (3)$$

here N is a nonlinear operator, $u(x,t)$ is an unknown function, and x and t denote space and time variables, respectively. By means of generalizing the traditional homotopy method, Liao[9] constructs the so called zero-order deformation equation:

$$(1-q)L[\varphi(x,t;q) - u_0(x,t)] = q\hbar H N[\varphi(x,t;q)], \qquad (4)$$

Where $q \in [01]$ is the embedding parameter, $\hbar$ a non-zero auxiliary parameter, H a non-zero auxiliary function, L an auxiliary linear operator, $u_0(x,t)$ an initial guess of $u(x,t)$ and $\varphi(x,t;q)$ is an unknown function. It is important that one has great freedom to choose auxiliary things in HAM. Obviously, when $q=0$ and $q=1$, it holds that

$$\varphi(x,t;0) = u_0(x,t), \qquad \varphi(x,t;1) = u(x,t). \qquad (5)$$

Thus, as q increases from 0 to 1, the solution $\varphi(x,t;q)$ varies from the initial guess $u_0(x,t)$ to the solution $u(x,t)$.

Expanding $\varphi(x,t;q)$ in Taylor series about $q=0$, we obtain

$$\varphi(x,t;q) = u_0(x,t) + \sum_{m=1}^{\infty} u_m(x,t)\, q^m, \qquad (6)$$

where

$$u_m(x,t) = \frac{1}{m!}\frac{\partial^m \varphi(x,t;q)}{\partial q^m}\Big|_{q=0}.$$

If the auxiliary linear operator, the initial guess, the nonzero auxiliary function and the nonzero auxiliary parameter are properly chosen, the above series converges at $q=1$, and then we obtain

$$\varphi(x,t;1) = u_0(x,t) + \sum_{m=1}^{\infty} u_m(x,t), \qquad (7)$$

which must be one of the solutions of the original non-linear differential equation.

Differentiating the zero-order deformation equation (4) m-times with respect to q, putting $q=0$ and finally dividing by $m!$, we obtain the mth-order deformation equation:

$$L[u_m(x,t) - \chi_m u_{m-1}(x,t)] = \hbar H R_m(u_{m-1}, x, t), \qquad (8)$$

where

$$R_m(u_{m-1}, x, t) = \frac{1}{(m-1)!}\frac{\partial^{m-1} N[\varphi(x,t;q)]}{\partial q^{m-1}}\Big|_{q=0},$$

And

$$\chi_m = \begin{cases} 0, & m \le 1 \\ 1, & m > 1 \end{cases}.$$

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INTRODUCTION

The second-order hyperbolic telegraph equation in one-space dimension, given by

$$\frac{\partial^2 u}{\partial t^2} + \alpha\frac{\partial u}{\partial t} + \beta u = \frac{\partial^2 u}{\partial x^2} + f(x,t), \qquad a \le x \le b,\ t \ge 0, \qquad (1)$$

with the initial conditions

$$\begin{cases} u(x,0) = g_1(x,t), & a \le x \le b \\ u_t(x,0) = g_2(x,t), & a \le x \le b \end{cases}, \qquad (2)$$

where α and β are known constant coefficients.

Equations of the form Eq. (1) arise in the study of propagation of electrical signals in a cable of transmission line and wave phenomena. Interaction between convection and diffusion or reciprocal action of reaction and diffusion describes a number of nonlinear phenomena in physical, chemical and biological process [1]-[4]. In fact the telegraph equation is more suitable than ordinary diffusion equation in modeling reaction diffusion for such branches of sciences. For example biologists encounter these equations in the study of pulsate blood flow in arteries and in one-dimensional random motion of bugs along a hedge [5]. Also the propagation of acoustic waves in Darcy-type porous media [6], and parallel flows of viscous Maxwell fluids [7] are just some of the phenomena governed [8,14] by Eq. (1). Recently various iterative methods are applied for getting Numerical and analytical solutions of telegraph equations [10,11,12,13].

Here the homotopy analysis method (HAM) is applied to solve the proposed equations. HAM introduced by Liao[9];the HAM is rather general and contains the Homotopy perturbation method (HPM) and the Adomian's decomposition method (ADM). The researchers have been successfully applying this method to various nonlinear problems in science and engineering [15-20].

Example 1: We consider the hyperbolic telegraph Eq. (1) with $\alpha = 6$ and $\beta = 2$ in the interval $0 \le x \le \pi$.

The initial conditions are given by [12,13]

$$\begin{cases} u(x,0) = \sin(x), & 0 \le x \le \pi \\ u_t(x,0) = -\sin(x) & 0 \le x \le \pi \end{cases}, \qquad (9)$$

The analytical solution is given in [12] as $u(x,t) = e^{-t}\sin(x)$ and

$$f(x,t) = (2 - \alpha + \beta)e^{-t}\sin x.$$

If the HAM is used for solving this problem then according to (9) we can choose the initial approximation,

$$u_0(x,t) = \sin(x) - t\sin(x)$$

and the linear operator

$$L[\varphi(x,t;q)] = \frac{\partial^2 \varphi(x,t;q)}{\partial t^2}$$,with the property $L[c_1 + c_2 t] = 0$

where c_1 is constant of integration. Here we have the zero-order deformation equation,

$$(1-q)L[\varphi(x,t;q) - u_0(x,t)] = q\hbar N[\varphi(x,t;q)] \qquad (10)$$

Where

$$N[\varphi(x,t;q)] = \frac{\partial^2 \varphi(x,t;q)}{\partial t^2} + \frac{\partial \varphi(x,t;q)}{\partial t} + \varphi(x,t;q) - \frac{\partial^2 \varphi(x,t;q)}{\partial x^2} - f(x,t)$$

Differentiating the zero-order deformation equations (13) m-times with respect to q ,putting $q=0$ and finally dividing by $m!$, we obtain the mth-order deformation equations:

$$L[u_m(x,t) - \chi_m u_{m-1}(x,t)] = \hbar H\, R_m(u_{m-1}(x,t)), \qquad (11)$$

with the initial conditions $u_m(x,0) = 0$. Where

$$R_m(u_{m-1}(x,t)) = N[u_{m-1}(x,t)]$$

For simplicity ,we choose $H = 1$ and $\hbar = h$

Using (11), we obtain the various approximations as follows:

$$u_0(x,t) + u_1(x,t) = \sin(x) - 2h\sin(x) + 2e^t\sin(x) + 2ht\sin(x) + \frac{3}{2}ht^2\sin(x) - \frac{1}{2}ht^3\sin(x) \qquad (12)$$

CONVERGENCE ANALYSIS

The eleven-term approximate series solutions are given by

$$u(x,t) = \sum_{i=0}^{10} u_i(x,t)$$

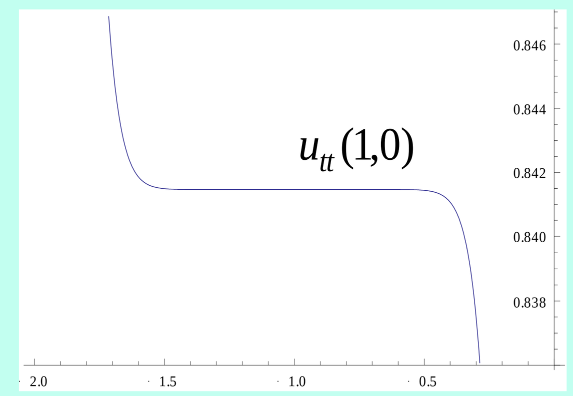

Fig. 1: The *h*-curves of $u_{tt}(1,0)$ with the initial condition (9) obtained by the eleven-term approximation of the HAM.

The convergence region may be taken where the *h*-curve is parallel or nearly parallel to *x*-axis. We have chosen $h = -1$ as this is most appropriate value for this problem

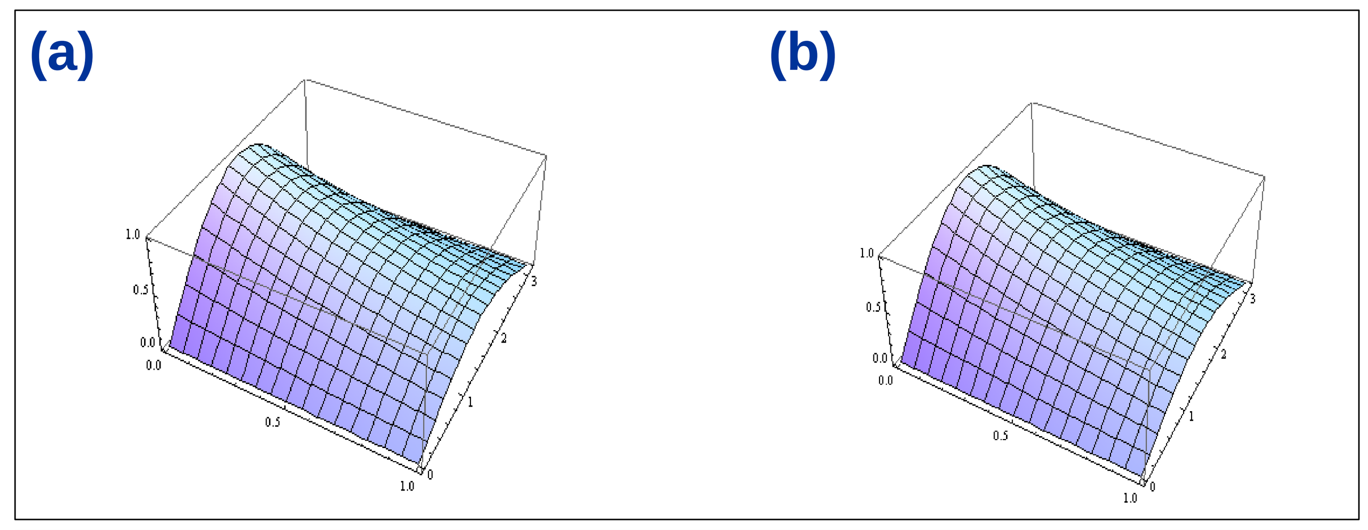

Fig. 2: (a) computed u(x,t) and (b) exact u(x,t)

Example 2: We consider the hyperbolic telegraph Eq. (1) with $\alpha = 1$ and $\beta = 1$ in the interval $0 \le x \le 3$.

The initial conditions are given by [12]

$$\begin{cases} u(x,0) = e^x, & 0 \le x \le 3 \\ u_t(x,0) = -e^x & 0 \le x \le 3 \end{cases}, \qquad (13)$$

and the analytical solution is given in [12] as

$$u(x,t) = \exp(x-t)$$

In this case $f(x,t) = 0$.

If the HAM is used for solving this problem then according to (13) we can choose the initial approximation

$$u_0(x,t) = e^x - te^x \qquad (14)$$

Using (11),we obtain the various approximation as fallows

$$u_0(x,t) + u_1(x,t) = e^x - e^x t - \frac{1}{2}(e^x h e^t)$$

$$u_0(x,t) + u_1(x,t) + u_2(x,t) = e^x - e^x t - \frac{1}{2}(e^x h e^t) + \frac{1}{6}(e^x h^2 t^3) \qquad (15)$$

h-curve: We have chosen $h = -1$ as this is most appropriate value for this problem.

The exact solution is given by

$$u(x,t) = \sum_{i=0}^{\infty} u_i(x,t) = e^{(x-t)} \qquad (16)$$

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