BRYAN'S EFFECT AND NONLINEAR DAMPING

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I. Introduction

If a node of vibration is marked on an annular plate (see Fig. 1) by a spot of paint and then a small inertial rotation rate $\varepsilon\Omega$ occurs about the z-axis, the node will be seen to move away from the spot. This effect was first observed in 1890 by G.H. Bryan [1] who calculated:

$$\text{Rate of rotation of the vibrating pattern} \;=\eta\varepsilon\Omega. \tag{1}$$

Consequently, if an ideal vibratory gyroscope is fixed inside a vehicle, the rate of rotation of the vibrating pattern (observed within the vehicle) may be used to determine the inertial rotation rate, once Bryan's factor η is known [2]. Referring to Fig. 1, where we use a

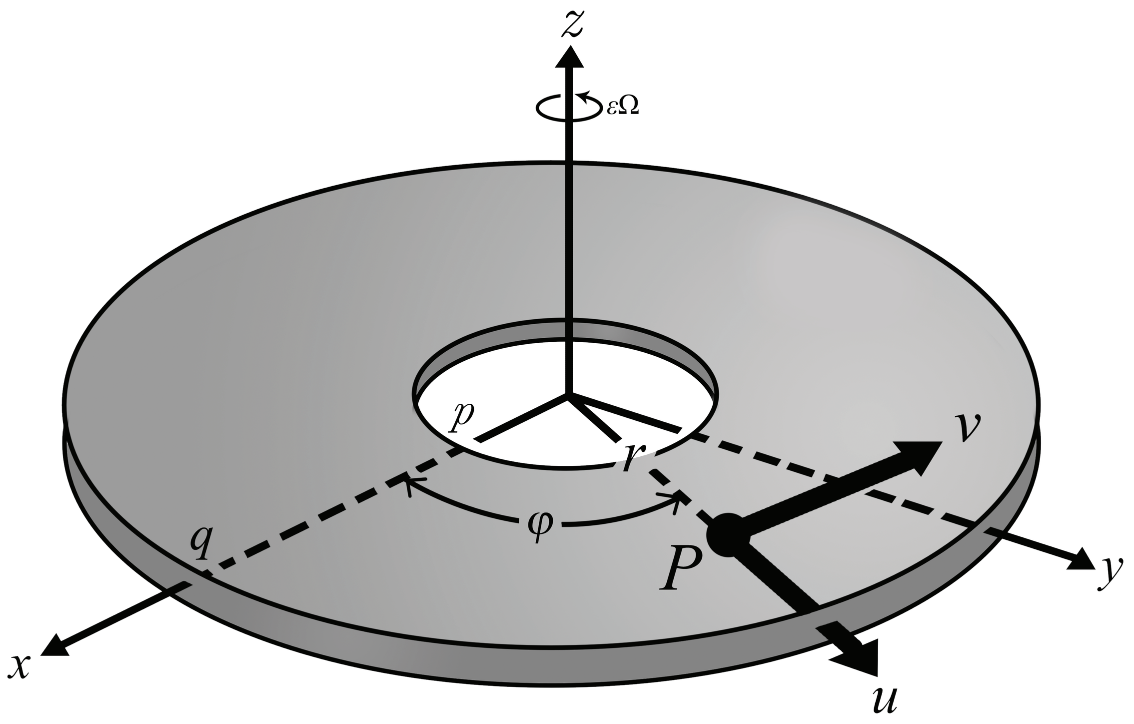

Fig. 1. The radial and tangential displacements u and v (from rest position P) respectively of a point mass in the plate.

polar coordinate system, we assume that the displacements satisfy:

$$u(r,\varphi,t) = U(r)\left[C(t)\cos m\varphi + S(t)\sin m\varphi\right] \tag{2}$$
$$v(r,\varphi,t) = V(r)\left[C(t)\sin m\varphi - S(t)\cos m\varphi\right] \tag{3}$$

where m is the circumferential wave number, $C(t)$ and $S(t)$ determine the principal $P(t)$ and quadrature $Q(t)$ vibration amplitudes and $U(r)$ and $V(r)$ are eigenfunctions associated with an eigenvalue ω of the vibration pattern.

II.DAMPING

Any anisotropic vibration damping causes a departure from the ideal distribution and therefore affects resonator dynamics. It can be shown, as was done for a sphere in [2], that the Lagrangian

$$L = \pi(\dot{C}^2+\dot{S}^2)I_0 + 2\varepsilon\Omega(\dot{C}S - C\dot{S})I_1 - \pi\left(C^2+S^2\right)I_3 \tag{4}$$

where I_0,I_1 and I_2 are constants determined by indefinite integrals of U and V, density ρ and Young's modulus E. We introduce anisotropic linear, quadratic, cubic, quartic damping with $n = 1,2,3,4$ respectively into the equations of motion using the Lagrange-Euler equations:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{C}}\right) - \frac{\partial L}{\partial C} = -\left|\dot{C}\right|^{k_n} \frac{\partial}{\partial \dot{C}}\mathcal{F}_n \tag{5}$$
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{S}}\right) - \frac{\partial L}{\partial S} = -\left|\dot{S}\right|^{k_n} \frac{\partial}{\partial \dot{S}}F_n \tag{6}$$

where $k_p = 0,1$ if p is odd, even respectively and the *modified Rayleigh dissipation function* is

$$F_n = \frac{h}{(n+k_{n+1})}\int_0^{2\pi}\int_p^q \Delta_n(\varphi)((\dot{u}^{n+k_{n+1}} + \dot{v}^{n+k_{n+1}})r\,dr\,d\varphi \tag{7}$$

where the *light damping coefficient* $\Delta_n(\varphi)$ is the sum of its Fourier series zeroth, $2m^{th}$ and $4m^{th}$ harmonics:

$$\Delta_n(\varphi) = \varepsilon\rho\Big(\tilde{\delta}_{n,0,c} + \tilde{\delta}_{n,2,c}\cos 2m\varphi + \tilde{\delta}_{n,2,s}\sin 2m\varphi + \tilde{\delta}_{n,4,c}\cos 4m\varphi + \tilde{\delta}_{n,4,s}\sin 4m\varphi\Big). \tag{8}$$

As explained in [2], regard $C(t)$ as the "cosine" and $S(t)$ as the "sine" output of a vibratory gyroscope connected to a two-channel oscilloscope. Then the Lissajous figure produced on the oscilloscope screen will resemble the precessing, shrinking, ellipse-like orbit depicted in Fig. 2. Using the methods of [2], we obtain,

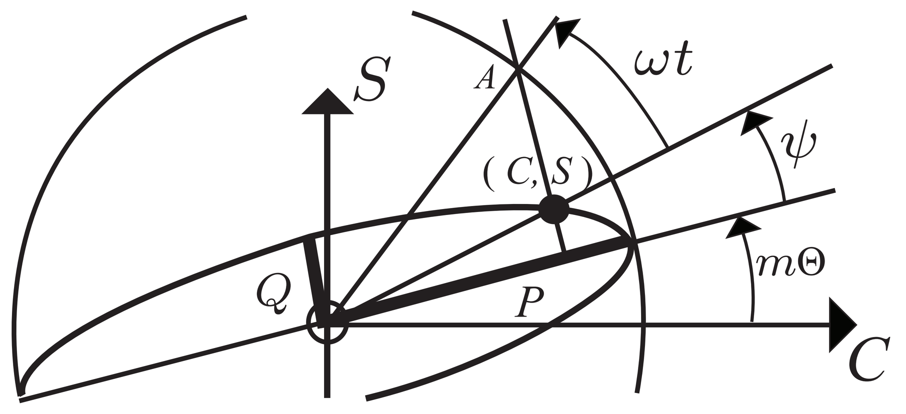

Fig. 2. $P(t)$ is the amplitude of the principal vibration, $Q(t)$ is the amplitude of the quadrature vibration, $m\Theta(t)$ is the rotation angle of the mth mode vibration pattern (the precession angle), $\psi(t)$ is a phase angle and $\omega = 2\pi/f$ where f is an eigenfrequency of the vibration pattern

for instance, the following good approximate equations of motion for cubic damping:

$$\dot{P} \approx -\tfrac{3}{128}\varepsilon\omega^2\left\{(6\delta_{3,0,c} + 4\Delta_2 + \Delta_4)P^3 + (2\delta_{3,0,c} - \Delta_4)PQ^2\right\} \tag{9}$$
$$\dot{Q} \approx -\tfrac{3}{128}\varepsilon\omega^2\left\{(6\delta_{3,0,c} - 4\Delta_2 + \Delta_4)Q^3 + (2\delta_{3,0,c} - \Delta_4)P^2Q\right\} \tag{10}$$
$$m\dot{\Theta} \approx \eta\varepsilon\Omega + \tfrac{3}{128}\varepsilon\omega^2\left(2\Delta_2'\tfrac{P^2+Q^2}{P^2-Q^2} + \Delta_4'\right)(P^2+Q^2) \tag{11}$$
$$\dot{\psi} \approx -\tfrac{3}{64}\varepsilon\omega^2\left(2\Delta_2'\tfrac{P^2+Q^2}{P^2-Q^2} + \Delta_4'\right)PQ \tag{12}$$

where Δ_2 and Δ_4 are (proportional to) the $2m^{th}$ and $4m^{th}$ harmonics respectively, while $\Delta_2' = \frac{-1}{2m}\frac{d\Delta_2}{d\varphi}$ and $\Delta_4' = \frac{-1}{2m}\frac{d\Delta_4}{d\varphi}$ with $\tilde{\delta}_{3,0,c} \propto \tilde{\delta}_{3,0,c}$ etc. and

$$\Delta_2 = \delta_{3,2,c}\cos 2m\varphi + \delta_{3,2,s}\sin 2m\varphi \tag{13}$$
$$\Delta_4 = \delta_{3,4,c}\cos 4m\varphi + \delta_{3,4,s}\sin 4m\varphi. \tag{14}$$

Evidently isotropic damping ($\delta_{3,0,c}$) affects only P and Q whereas anisotropic damping causes changes in all four variables as well as yielding different decay rates for P and Q. Equation (11) confirms Bryan's Equation (1) for the ideal case and shows that the precession rate $m\dot{\Theta}$ deviates from the ideal situation when anisotropic damping is present, as demonstrated by a numerical experiment conducted with Mathematica's NDSolve routine that resulted in Fig. 3. In order to eliminate anisotropic damping (but not necessarily nonlinear damping) materials with a high Q-factor must be used.

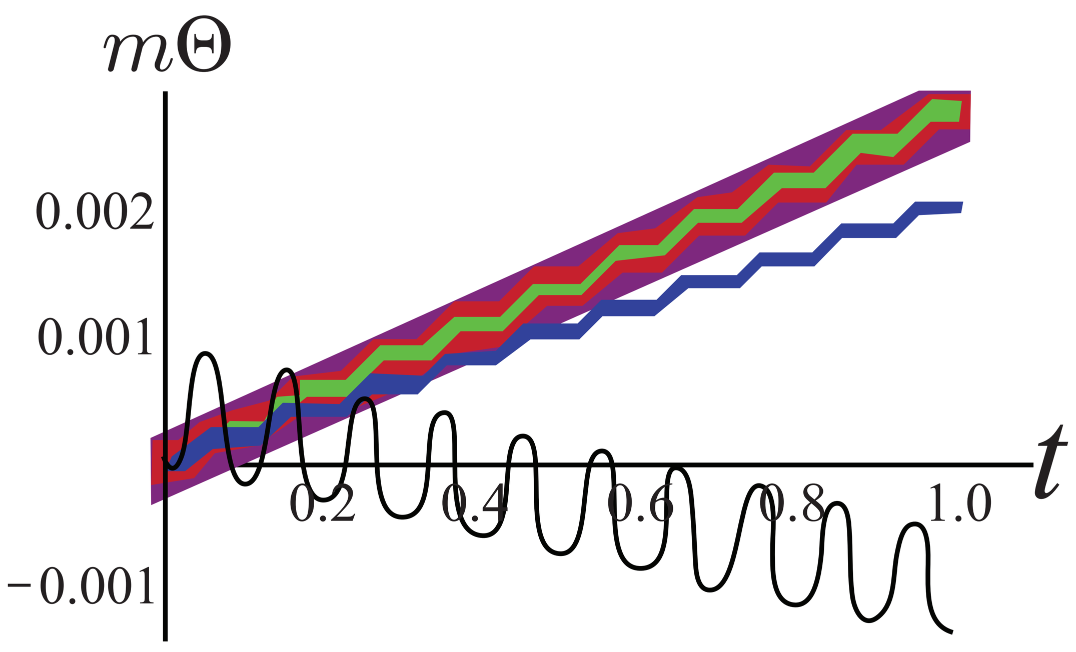

Fig. 3. The blue and black curves represent the change with time of the precession angle $m\Theta$ for cubic and quartic anisotropic damping respectively that show significant deviation from the ideal situation, that is, the solid purple line $m\Theta = \eta\varepsilon\Omega t$, while the green and red wavy curves (superimposed upon the purple line) represent $m\Theta$ for linear and quadratic anisotropic damping respectively.

References

[1] G.H. Bryan, "On the beats in the vibrations of a revolving cylinder or bell". *Proc. Cambridge Philo. Soc.* **7**, (1890), 101-111.

[2] M.Y. Shatalov, S.V. Joubert, C.E. Coetzee and I. Fedotov, "The influence of mass imperfections on the evolution of standing waves in slowly rotating spherical bodies". *JSV*, 330, (2011), 127-135.