Critical Thresholds on Presure-less Navier-Stokes equations with nonlocal viscosity

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N-S equations with nonlocal viscosity

$$\rho_t + div\,(\rho\mathbf{u}) = 0, \qquad \mathbf{x}\in\mathbb{R}^n, t\geq 0, \qquad (1)$$
$$\mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla p = \int_{\mathbb{R}^n} a(\mathbf{x},\mathbf{y})(\mathbf{u}(\mathbf{y})-\mathbf{u}(\mathbf{x}))\rho(\mathbf{y})d\mathbf{y},$$

- Initial Conditions: $\rho(\mathbf{x},0)=\rho_0(\mathbf{x})\geq 0, \quad \mathbf{u}(\mathbf{x},0)=\mathbf{u}_0(\mathbf{x})$.
- Pressure: $p=p(\rho)=\rho^\gamma$. $\gamma>0$: isentropic, $\gamma=0$: pressure-less.

Motivation: self-organized dynamics

- Self-organized dynamics: modeling the motion of self-propelled particles


A school of fish


A flock of birds

- N-particle flocking models:
$$\dot{x}_i = v_i, \quad \dot{v}_i = \sum_{j=1}^N a(\mathbf{x}_j,\mathbf{x}_i)(v_j-v_i).$$
- Macroscopic flocking: a hydrodynamics approach yields system (1).
  - Diameter of $supp(\rho)$ is uniformly bounded all time.
  - Maximum variation of $\mathbf{u}$ vanishes in time. *Fast* flock if it decays exponentially.
- **Cucker-Smale model** (Symmetric kernel):
$$a(\mathbf{x},\mathbf{y}) = \phi(\mathbf{x}-\mathbf{y}),$$
- **Motsch-Tadmor model** (Asymmetric kernel):
$$a(\mathbf{x},\mathbf{y}) = \frac{\phi(\mathbf{x}-\mathbf{y})}{\int_{\mathbb{R}^n}\phi(\mathbf{x}-\mathbf{y})\rho(\mathbf{y})d\mathbf{y}}.$$

Symmetric interaction kernel

For macroscopic Cucker-Smale model, there are *three different regimes* of interest depending on the choice of the kernel ϕ.

1. Local system.

- Hyperbolic scaling
$$(\mathbf{x},t)\to\left(\frac{\mathbf{x}}{\epsilon},\frac{t}{\epsilon}\right), \quad \phi\to\phi_\epsilon := \frac{1}{\epsilon^n}\phi\left(\frac{\mathbf{x}}{\epsilon}\right), \quad \epsilon\to 0.$$
- Compressible Navier-Stokes equations with degenerate viscosity coefficient.
$$\rho_t + div\,(\rho\mathbf{u}) = 0,$$
$$(\rho\mathbf{u})_t + div\,(\rho\mathbf{u}\otimes\mathbf{u}) + \nabla P = div\,\left(\rho^2\nabla\mathbf{u}\right).$$
- All known results are attempting to "avoid" *vacuum*.

2. Fractional dissipation.

- Kernel ϕ has a singularity at the origin.
- A typical prototype:
$$\phi(\mathbf{x}) = |\mathbf{x}|^{-n-2\alpha}.$$
- Widely discussed for $\rho\equiv 1$ in 1D (fractional dissipation).
$$\mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u} = -(-\Delta)^\alpha\mathbf{u}.$$
- Few discussions on full compressible system.

3. Nonlocal means.

- Kernel ϕ is bounded and continuous at the origin.
- A typical prototype:
$$\phi(\mathbf{x}) = (1+|\mathbf{x}|)^{-\alpha}.$$
- Slow decay rate at infinity: $\alpha\in(0,1)$ – Strong non-locality.

Assumptions and *main* system

Assumptions

- *Pressure-less.* Flocking property is preserved.
- *Bounded density.* ρ_0 is compactly supported.
- *Nonlocal means.* The interaction kernel ϕ is radial, decreasing and differentiable in r, satisfying
  - **(H1) Bounded at origin.** $\phi(0)=\|\phi\|_{L^\infty}=1$.
  - **(H2) Slow decay at infinity.** $\int^\infty \phi(r)dr = \infty$.

Main System

$$\rho_t + div\,(\rho\mathbf{u}) = 0, \qquad \mathbf{x}\in\mathbb{R}^n, t\geq 0, \qquad (2)$$
$$\mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u} = \int_{\mathbb{R}^n}\phi(\mathbf{x}-\mathbf{y})(\mathbf{u}(\mathbf{y})-\mathbf{u}(\mathbf{x}))\rho(\mathbf{y})d\mathbf{y},$$

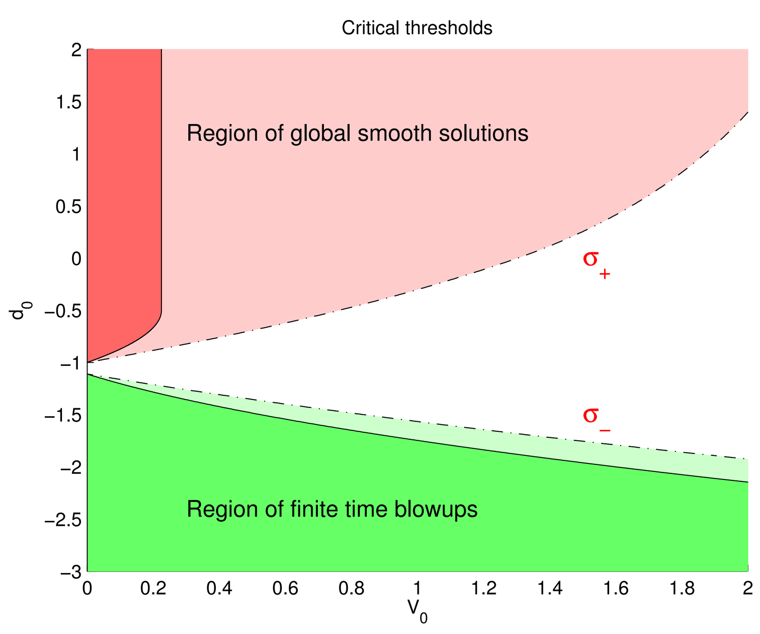
- IC: ρ_0 compact supported, $\mathbf{u}_0$ uniformly bounded.

Main theorem: Existence of smooth solution

- Critical threshold phenomenon for system (2):
  - There exists an *upper threshold* σ_+, depending on the initial profile, above which yields global smooth solution.
  - There exists a *lower threshold* σ_-, depending on the initial profile, below which yields finite time break down of smooth solution.
- Initial quantities of the velocity field which play a significant role in determining critical thresholds:
  - Max variation: $V_0 := \sup_{\mathbf{x},\mathbf{y}\in supp(\rho_0)} |\mathbf{u}_0(\mathbf{x})-\mathbf{u}_0(\mathbf{y})|$.
  - Min divergence: $d_0 := \inf_{\mathbf{x}\in supp(\rho_0)} div\,\mathbf{u}_0(\mathbf{x})$.

Theorem 1 [1D Critical Thresholds]. Consider system (2) in 1D with smooth initial data $\rho_0\in H^s_+(\mathbb{R})$ and $u_0\in H^{s+1}(supp(\rho_0))$, where $s>1$, satisfying all assumptions. Then,

- There exists a function σ_+ such that, if $d_0>\sigma_+(V_0)$, then there exists a global smooth solution $\rho\in C(\mathbb{R}^+,H^s_+(\mathbb{R}))$ and $u\in C(\mathbb{R}^+,H^{s+1}(supp(\rho)))$.
- There exists a function σ_- such that, if $d_0<\sigma_-(V_0)$, then the smooth solution (ρ,u) will break down in finite time.



- Illustration of the thresholds
  - Expressed in terms of (V_0,d_0).
  - Larger area of thresholds with fast flock.
  - A gap between the two thresholds due to L^∞ estimates, which is *not* sharp.
- 2D critical thresholds
  One more initial quantity of the velocity field needs to be bounded to guarantee existence of upper threshold
$$B_0 := \sup_{\mathbf{x}\in supp(\rho_0)} \max\left\{2|\partial_{x_1}u_{02}|, 2|\partial_{x_2}u_{01}|, |\partial_{x_1}u_{01}-\partial_{x_2}u_{02}|\right\}.$$

Theorem 2 [2D smooth solutions]. Smooth solution exists in 2D if initially
$$d_0>\sigma_+(V_0), \quad B_0<\zeta(V_0).$$

Key aspects of the proof

Smoothness via boundedness of $\nabla\mathbf{u}$.

Lemma. Consider system (2) in 1D or 2D with initial data $\rho_0\in H^s(\mathbb{R}^n)$ and $\mathbf{u}_0\in H^{s+1}(\mathbb{R}^n)$, where $s>1$. Then the following statements are equivalent.

- There exists a unique solution $(\rho,\mathbf{u})\in C([0,T];H^s)\times C([0,T];H^{s+1})$,
- $\|\nabla\mathbf{u}(\cdot,t)\|_{L^\infty}$ is bounded for all $t\in[0,T]$.

Dynamics of $div\,\mathbf{u}$ along particle path.

Let $d=div\,\mathbf{u}$. Applying ∇_x operator on (2), we get
$$d' = -d^2-(\phi\star\rho)d+\int_{\mathbb{R}^n}\nabla_x\phi(\mathbf{x}-\mathbf{y})\cdot(\mathbf{u}(\mathbf{y})-\mathbf{u}(\mathbf{x}))\rho(\mathbf{y})d\mathbf{y}+b.$$
where $'=\partial_t+\mathbf{u}\cdot\nabla_x$ and $b=\mathrm{tr}((\nabla\mathbf{u})^2)-(div\,\mathbf{u})^2$.
In 1D, $b=0$.
With bounded coefficients, this is a Reccatti-type system of d. It yields a critical threshold phenomenon.

Controlling other terms of $\nabla\mathbf{u}$.

In 2D, in addition to d, it is also needed to control other terms of $\nabla\mathbf{u}$, i.e. B. The following lemma shows that B is bounded if d is not too negative.

Lemma. Suppose $B(0)$ is bounded and $d(t)\geq-\delta_0$ for $t\in[0,T]$, where δ_0 is positive and determined by initial conditions. Then $B(t)$ has the same bound for $t\in[0,T]$.

Coupling with propagation of d yields similar critical threshold phenomenon.

Dynamics inside the vacuum.

Taking advantage of the slow decay of ϕ, we can extend $\mathbf{u}_0$ to the whole space such that the solution $\mathbf{u}$ inside the vacuum remains smooth all time.

The criterions of the extension read (e.g. in 1D)

- *Boundedness*: For all $\mathbf{x}\in\mathbb{R}^n$,
$$\min_{\mathbf{y}\in supp(\rho_0)}\mathbf{u}_0(\mathbf{y})\leq\mathbf{u}_0(\mathbf{x})\leq\max_{\mathbf{y}\in supp(\rho_0)}\mathbf{u}_0(\mathbf{y}),$$
- *Uniform limit*: $\lim_{|\mathbf{x}|\to\infty}\mathbf{u}_0(\mathbf{x})=\mathbf{c}_\infty$.
- *Avoid fast decay*:
$$\partial_x u_0(x)\geq-\frac{m}{2}\phi(L(x,0)+D), \quad \text{for } x\notin supp(\rho_0),$$
where $L(x,0)=dist(x,supp(\rho_0))$ and $D>0$.

Extension and limitation

- This technique can be also used to Mostch-Tadmor model, with slightly different threshold functions.
- The boundedness and Lipschitz property of ϕ is needed to control $div\,\mathbf{u}$. When passing to the hyperbolic limit, $\|\phi\|_{Lip}$ blows up. Therefore, the argument is *not* true for local system.

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Acknowledgment: This research was supported in part by NSF grants KI-NET, RNMS #11-07444 and DMS #10-08397.

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