ON HYPERBOLIC EQUATIONS OF THE PROBLEM OF LONGITUNDINAL VIBRATIONS OF GROWING RODS

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ABSTRACT

Dynamics of growing bodies is a new and rapidly developing area of mechanics and applied mathematics. In the present paper, the longitudinal forced and free longitudinal vibrations of a slender rod are considered. It is assumed that in the process of growing the rod, its area is constant and length is increased in accordance with a prescribed law, and hence depends on time. The rod material is assumed to be elastic and isotropic. The dynamics of the rod are described by equations of the linear elasticity. The main hyperbolic equations of the growing rod are derived from the classical Rayleigh-Love and Rayleigh-Bishop models. The qualitative effects of the growing longitudinally vibrating rod are discussed: proportional growth of amplitude of vibrations at linear increasing of the rod length and excitation of the neighbour modes in the process of vibration and growth.

EQUATIONS AND BOUNDARY CONDITION OF THE ROD DESCRIBED BY THE CLASSICAL MODEL

In the classical case, the longitudinal motions of the vibrating rod (its dynamics) are described by the wave equation:

$$\frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2} = F(t,x) \tag{1}$$

where $u = u(t,x)$ is longitudinal displacement of the rod, $F(t,x)$ is exciting force and $c^2 = E/\rho$ is the square of the speed of the wave propogation. It is assumed that the left end of the rod is fixed and its right end is free. The process of the rod growth is realized by means of deposition of the material of the rod on its right end. Hence, the boundary conditions are:

$$x = 0: \qquad u(t,0) = 0$$
$$x = 1+\varepsilon\cdot f(t): \quad \frac{\partial u}{\partial x}\left(t, 1+\varepsilon\cdot f(t)\right) = 0 \tag{2}$$

where ε is a small parameter proportional to the speed of growth of the rod.

To represent the boundary value problem (1) - (2) in the standard form it is necessary to use transformation $(t,x) \to (\tau,y)$:

$$t=\tau; \qquad x = y\cdot[1+\varepsilon\cdot f(t)] \tag{3}$$

In new parameters (τ,y) equation (1) is as follows:

$$\frac{\partial^2\tilde u}{\partial\tau^2} - \frac{2\varepsilon y f'(\tau)}{1+\varepsilon\cdot f(\tau)}\frac{\partial^2\tilde u}{\partial\tau\partial y} - \frac{c^2-\varepsilon^2y^2}{[1+\varepsilon\cdot f(\tau)]^2}\frac{\partial^2\tilde u}{\partial y^2} - \left\{\frac{\varepsilon y f''(\tau)}{1+\varepsilon\cdot f(\tau)} - \frac{2\varepsilon^2 y[f'(\tau)]^2}{[1+\varepsilon\cdot f(\tau)]^2}\right\}\frac{\partial\tilde u}{\partial y} = \tilde F(\tau,y) \tag{4}$$

where

$$f'(\tau) = \frac{df(\tau)}{d\tau},\quad f''(\tau) = \frac{d^2f(\tau)}{d\tau^2},\quad \tilde u(\tau,y) = u\left[t=\tau, x = y\cdot(1+\varepsilon\cdot f(\tau))\right],$$
$$\tilde F(\tau,y) = F\left[t=\tau, x=y\cdot(1+\varepsilon\cdot f(\tau))\right].$$

In this case boundary conditions (2) are:

$$y=0:\ \tilde u(\tau,0)=0; \qquad y=1:\ \frac{\partial\tilde u}{\partial y}(\tau,1)=0 \tag{5}$$

In the particular case $f(\tau)=\tau$ (linear growth of the rod) equation (4) is:

$$A^{(Cl)}\tilde u(\tau,y) = \tilde F(\tau,y) \tag{6}$$

where the classical linear differential hyperbolic operator is:

$$A^{(Cl)} = \frac{\partial^2}{\partial\tau^2} - \frac{2\varepsilon y}{1+\varepsilon\tau}\frac{\partial^2}{\partial\tau\partial y} - \frac{c^2-\varepsilon^2y^2}{(1+\varepsilon\tau)^2}\frac{\partial^2}{\partial y^2} + \frac{2\varepsilon^2 y}{(1+\varepsilon\tau)^2}\frac{\partial}{\partial y} \tag{7}$$

THE RAYLEIGH-LOVE MODEL OF VIBRATING ROD

Original equation of the vibrating rod in this model is:

$$\frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2} - a^2\frac{\partial^4 u}{\partial t^2\partial x^2} = F(t,x) \tag{8}$$

where $a^2 = \nu^2 I_p / S$, ν is the Poisson ratio, I_p is the polar moment of inertia and S is the area of the cross-section of the rod.

Transforming equation (8) by (3) we obtain:

$$\left[A^{(Cl)} - a^2A^{(R-L)}\right]\tilde u(\tau,y) = \tilde F(\tau,y) \tag{9}$$

where

$$A^{(R-L)} = \frac{1}{(1+\varepsilon\tau)^2}\left\{\frac{\partial^4}{\partial\tau^2\partial y^2} - \frac{2\varepsilon}{1+\varepsilon\tau}\left[y\frac{\partial^4}{\partial\tau\partial y^3} + 2\frac{\partial^3}{\partial\tau\partial y^2}\right] + \frac{\varepsilon^2}{(1+\varepsilon\tau)^2}\left[y^2\frac{\partial^4}{\partial y^4} + 6y\frac{\partial^3}{\partial y^3} + 6\frac{\partial^2}{\partial y^2}\right]\right\} \tag{10}$$

THE RAYLEIGH-BISHOP MODEL OF VIBRATING ROD

Original equation of the vibrating rod in this model is:

$$\frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2} - a^2\frac{\partial^4 u}{\partial t^2\partial x^2} + b^2\frac{\partial^4 u}{\partial x^4} = F(t,x) \tag{11}$$

where

$b^2 = \nu^2 G I_p / \rho S$, G is the shear modulus of elasticity and ρ is the mass density of the rod.

Transforming equation (11) by (3) we obtain:

$$\left[A^{(Cl)} - a^2A^{(R-L)} + b^2A^{(R-B)}\right]\tilde u(\tau,y) = \tilde F(\tau,y) \tag{12}$$

where

$$A^{(R-B)} = \frac{1}{(1+\varepsilon\tau)^4}\frac{\partial^4}{\partial y^4} \tag{13}$$

MODELLING EQUATION FOR THE VIBRATING ROD

The modelling equation is

$$\frac{d^2W(\tau)}{d\tau^2} - \frac{\varepsilon}{1+\varepsilon\tau}\frac{dW(\tau)}{d\tau} + \frac{\omega^2}{(1+\varepsilon\tau)^2}W(\tau) = 0 \tag{14}$$

has general solution:

$$W(\tau) = (1+\varepsilon\tau)\left\{C_1\cos[\alpha\cdot\ln(1+\varepsilon\tau)] + C_2\sin[\alpha\cdot\ln(1+\varepsilon\tau)]\right\} \tag{15}$$

where $\alpha = \sqrt{\left(\omega/\varepsilon\right)^2 - 1}$

Hence, amplitude of solution (15) grows proportionally to time τ.

NUMERICAL ANALYSIS

The numerical solution is obtained using the Galerkin method with base functions

$$\sin\left[\frac{(2k+1)\pi}{2}y\right],$$ satisfying BC (5).

The following representation of solution is used:

$$\tilde u(\tau,y) \approx \sum_{k=0}^{N} C_k(\tau)\sin\left[\frac{(2k+1)\pi}{2}y\right] \tag{16}$$

where $C_k(\tau)$ are unknown functions of time.

For initial conditions:

$$\tau=0:\quad \tilde u(0,y) = \sin\left(\frac{\pi y}{2}\right), \qquad \frac{\partial\tilde u}{\partial\tau}(0,y) = 0 \tag{17}$$

solution of equation (6) - (7) of the classical model for the first four modes is shown in Fig. 1.

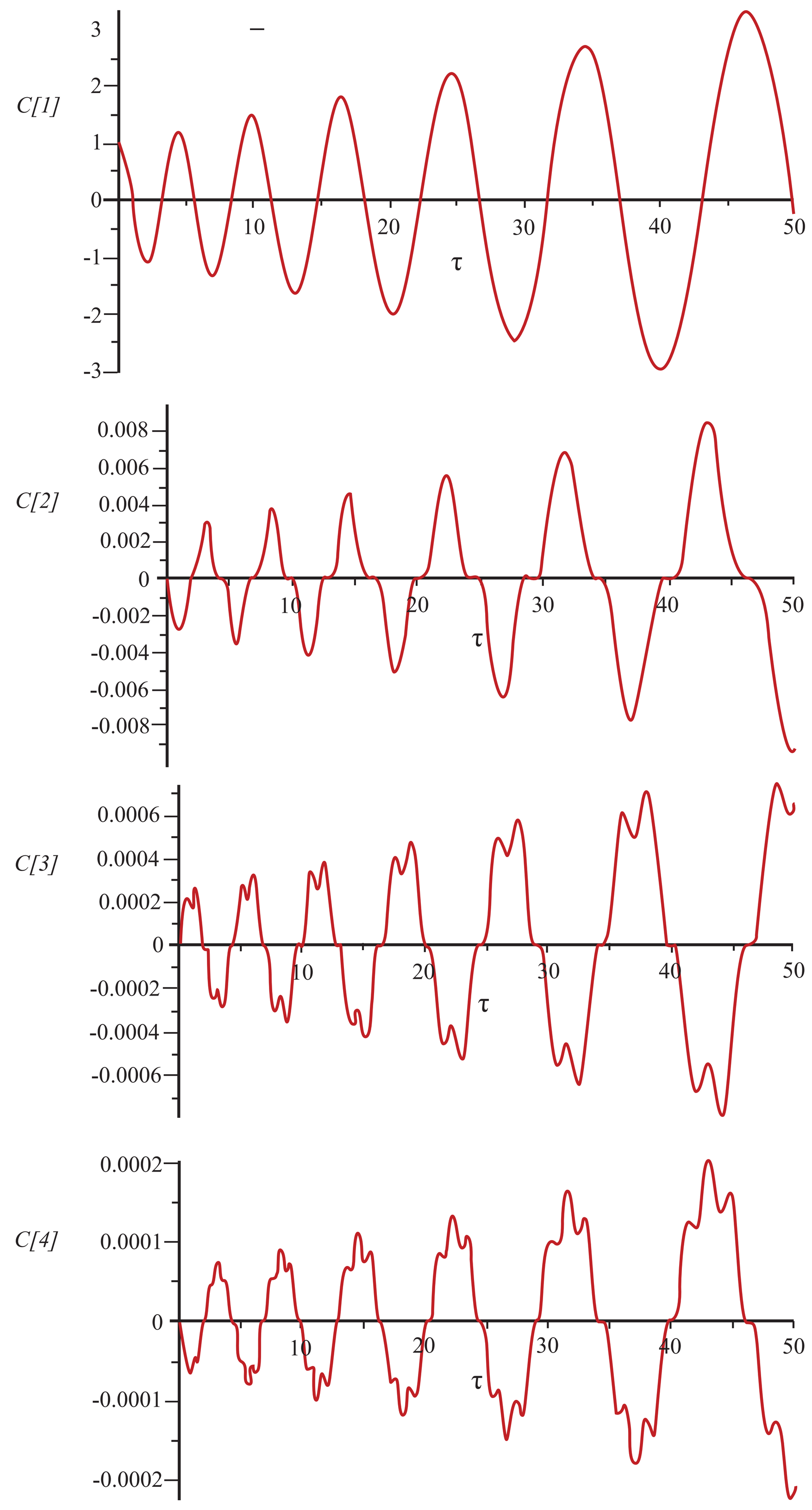

Fig. 1.

Note that the graph of solution of modelling equation (16) with initial conditions $W(0)=1,\ \frac{dW}{d\tau}(0)=0$ is visually indistinguishable from the first plot in Fig. 1.

Results of numerical analysis of equations (8) - (9) and (11) - (12) of the Rayleigh-Love and Rayleigh-Bishop models demonstrate the qualitative similarity with the classical model.