Boundary Controllability of a 1D Flow Model with ODE Boundary Conditions

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Nonlinear and Linear Model

(Nonlinear)

Continuity Equation

$$\frac{\partial A}{\partial t} = -u\frac{\partial A}{\partial x} - A\frac{\partial u}{\partial x}$$

Momentum Equation

$$\frac{\partial u}{\partial t} = -u\frac{\partial u}{\partial x} - \frac{sE}{2\rho r_0}(A_0 A)^{-1/2}\frac{\partial A}{\partial x} - \frac{8\pi\mu}{\rho A_0}u$$

Evolution of Level Heights

$$\frac{dh_0}{dt} = -\frac{1}{A_T}(Au)|_{x=0}$$

$$\frac{dh}{dt} = \frac{1}{A_T}(Au)|_{x=\ell}$$

(BCs)

$$A|_{x=0} = A_0\left(1 + \frac{r_0}{sE}(\rho g h_0 + p_{f0})\right)^2$$

$$A|_{x=\ell} = A_0\left(1 + \frac{r_0}{sE}(\rho g h + p_f)\right)^2$$

(ICs)

$$A|_{t=0} = A^0, \quad u|_{t=0} = u^0,$$

$$h_0|_{t=0} = h_0^0, \quad h|_{t=0} = h^0$$

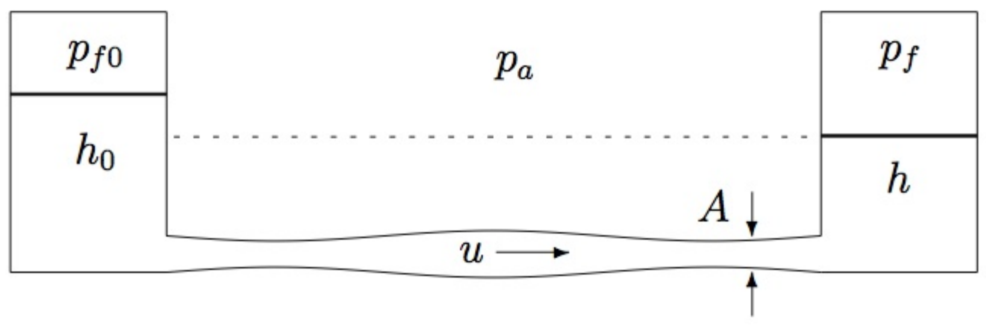

Figure: An elastic tube connected to two tanks

(Linearized)

$$\frac{\partial \tilde{A}}{\partial t} = -A_e\frac{\partial \tilde{u}}{\partial x}$$

$$\frac{\partial \tilde{u}}{\partial t} = -\alpha\frac{\partial \tilde{A}}{\partial x} - \beta\tilde{u}$$

$$\frac{d\tilde{h}_0}{dt} = -\frac{A_e}{A_T}\tilde{u}|_{x=0}$$

$$\frac{d\tilde{h}}{dt} = \frac{A_e}{A_T}\tilde{u}|_{x=\ell}$$

$$\tilde{A}|_{x=0} = \gamma\tilde{h}_0 + \tilde{p}_{f0}, \quad \tilde{A}|_{x=\ell} = \gamma\tilde{h} + \tilde{p}_f$$

$$\tilde{A}|_{t=0} = \tilde{A}^0, \quad \tilde{u}|_{t=0} = \tilde{u}^0, \quad \tilde{h}_0|_{t=0} = \tilde{h}_0^0, \quad \tilde{h}|_{t=0} = \tilde{h}^0$$

A (cross-section), u (velocity), h_0 and h (level heights of the left and right tanks), A_T (cross-section of tanks), A_0 (rest cross-section), r_0 (rest radius), ℓ (length of tube), s (thickness of tube), E (Young's modulus), ρ (density of the fluid), g (gravitational constant), μ (viscosity), $(A_e, 0, h_{0e}, h_e)$ is the equilibrium state, $\alpha, \gamma > 0$ and $\beta \geq 0$ (compound parameters), p_{f0} and p_f (forcing functions on the left and right tanks).

Results for the Linearized Model

- **Semigroup well-posedness** using Lumer-Philipps Theorem in the state space

$$X = L^2((0,\ell);\mathbb{C}) \times L^2((0,\ell);\mathbb{C}) \times \mathbb{C}^2.$$

The differential operator generates a $\mathcal{C}_0$-group, which is unitary in the absence of viscosity.

- Infinitesimal generator is a **discrete spectral operator** for system with small viscosity μ. The spectrum $(\lambda_n)_{n\in\mathbb{Z}}$ satisfies

$$\lambda_0 = 0, \qquad \lambda_{-n} = \overline{\lambda_n}$$

$$\lambda_n = -\frac{\beta}{2} + \left(\frac{(n-1)\pi c}{\ell} + \mathcal{O}(n^{-1})\right)i, \qquad n \to \infty$$

where $c = \sqrt{\alpha A_e}$ is the speed of propagation.

- **Uniform exponential stability** in the state space of "*deviations from the equilibrium*"

$$H := X_0^{\perp} \simeq X/X_0 = \ker(V)$$

for $\mu > 0$, where X_0 is the eigenspace associated with the zero eigenvalue and $V : X \to \mathbb{C}$ is the volume functional

$$V(A, u, h_0, h) = \int_0^\ell A(x)\,dx + A_T(h_0 + h).$$

Proof is based on Gearhart-Prüss Theorem.

- **Riesz basis** generation using the results of Zwart (2010), Xu-Guo (2003) and perturbation theory for spectral operators.

- Boundary input operator $\mathcal{B}$ is unbounded. Well-posedness of the control system is based on an extension of the generator.

- **Exact controllability** of the system (on the state space H) with small viscosity using the Fourier representation of the solution and Tucsnak-Weiss' Simultaneous Controllability Theorem for times $T > 2c$.

Control A at the boundary by $\mathcal{B}$ $\leftrightarrow$ Observe u at the boundary by $\mathcal{B}^*$

- **Critical time of controllability** $T^* = 2c$ via Kadec's 1/4 Theorem:

For single input controls, if $T < 2c$ then the system is not exactly controllable.

- **Characterization of Control**

Given $w_0, w_1 \in H_{\mathbb{R}}$ (real vector space) define $\mathcal{J}(\cdot; w_0, w_1) : H_{\mathbb{R}} \to \mathbb{R}$ by

$$\mathcal{J}((B, v, g_0, g); w_0, w_1) = \frac{1}{2}\int_0^T (|u(t,0)|^2 + |u(t,\ell)|^2)\,dt + (w_0, (B, v, g_0, g))_X - (w_1, (A, u, h_0, h)(T))_X$$

where $(A, u, h_0, h)(t) = e^{-\mathcal{A}^* t}(B, v, g_0, g)$.

THEOREM *If $z_0, z_1 \in H$ (complex vector space) and*

$$\zeta^\star = \underset{z\in H_{\mathbb{R}}}{\operatorname{argmin}}\,\mathcal{J}(z; \Re z_0, \Re z_1),$$

$$\vartheta^\star = \underset{z\in H_{\mathbb{R}}}{\operatorname{argmin}}\,\mathcal{J}(z; \Im z_0, \Im z_1)$$

then $u^\star(t) = \mathcal{B}^* e^{-\mathcal{A}^* t}(\zeta^\star + i\vartheta^\star)$ satisfies $(A, u, h_0, h)(T; z_0, u^\star) = z_1$ and is optimal in the L^2-sense:

$$\|u^\star\|_{L^2(0,T;\mathbb{C}^2)} \leq \|v\|_{L^2(0,T;\mathbb{C}^2)}$$

for all $v \in L^2(0, T; \mathbb{C}^2)$ such that $(A, u, h_0, h)(T; z_0, v) = z_1$.

References

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[2] G. Xu and B. Guo, *Riesz basis property of evolution equations in Hilbert spaces and application to a coupled string equation*, SIAM J. Control Optim. 42 (2003) pp. 966-984.

[3] H. Zwart, *Riesz basis for strongly continuous groups*, J. Differential Equations 249 (2010) pp. 2397-2408.

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