Zero dissipation limit with vacuum

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1. Introduction

There have been many results on the zero dissipation limit of the compressible fluid with basic wave patterns without vacuum. For the system of the hyperbolic conservation laws with artificial viscosity

$$u_t + f(u)_x = \varepsilon u_{xx},$$

Goodman-Xin [3] first verified the viscous limit for piecewise smooth solutions separated by non-interacting shock waves using a matched asymptotic expansion method. Later Yu [9] proved it for the corresponding hyperbolic conservation laws with both shock and initial layers. In 2005, important progress made by Bianchini-Bressan[1] justifies the vanishing viscosity limit in BV space even though the problem is still unsolved for the physical system such as the compressible Navier-Stokes(denoted by CNS) equations.
For the CNS equations (1), Hoff-Liu [4] first proved the vanishing viscosity limit for piecewise constant shock even with initial layer. Later Xin [8] obtained the zero dissipation limit for rarefaction waves without vacuum for both rarefaction wave data and well-prepared smooth data.
More recently, Chen-Perepelitsa [2] proved the vanishing viscosity to the compressible Euler equations for the CNS equations (1) by compensated compactness method for the case that the far field of the initial values of Euler system (2) has no vacuums.
Now we turn back to the case of the basic wave patterns with vacuum states. As pointed out by Liu-Smoller [6], among the two nonlinear waves, i.e., shock and rarefaction waves, to the one-dimensional compressible isentropic Euler equations (2), only the rarefaction wave can be connected to vacuum.
In this presentation, we investigate this fundamental problem and want to obtain the decay rate with respect to the viscosity ϵ. Remark that Perepelitsa [7] consider the time-asymptotic stability of solutions of 1-d CNS equations (1) toward rarefaction waves connected to vacuum in Lagrangian coordinate and Jiu-Wang-Xin [5] study the large time asymptotic behavior toward rarefaction waves for solutions to the 1-dimensional CNS equations (1) with density-dependent viscosity for general initial data whose far fields are connected by a rarefaction wave to the corresponding Euler equations with one end state being vacuum.
We investigate the zero dissipation limit of the one-dimensional compressible isentropic Navier-Stokes equations

$$\begin{cases} \rho_t + (\rho u)_x = 0, & x \in \mathbf{R}, t > 0, \\ (\rho u)_t + \big(\rho u^2 + p(\rho)\big)_x = \epsilon\, u_{xx}, \end{cases} \tag{1}$$

where $\rho(t,x) \geq 0$, $u(t,x)$ and p represent the density, the velocity and the pressure of the gas, respectively and $\epsilon > 0$ is the viscosity coefficient. Here we assume that the viscosity coefficient ϵ is a positive constant and the pressure p is given by the $\gamma-$law:

$$p(\rho) = \frac{\rho^\gamma}{\gamma}$$

with $\gamma > 1$ being the gas constant.
Formally, as ϵ tends to zero, the limit system of the CNS equations (1) is the following inviscid Euler equations

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + (\rho u^2 + p(\rho))_x = 0. \end{cases} \tag{2}$$

For definiteness, 2-rarefaction wave will be considered. If we investigate the compressible Euler system (2) with the Riemann initial data

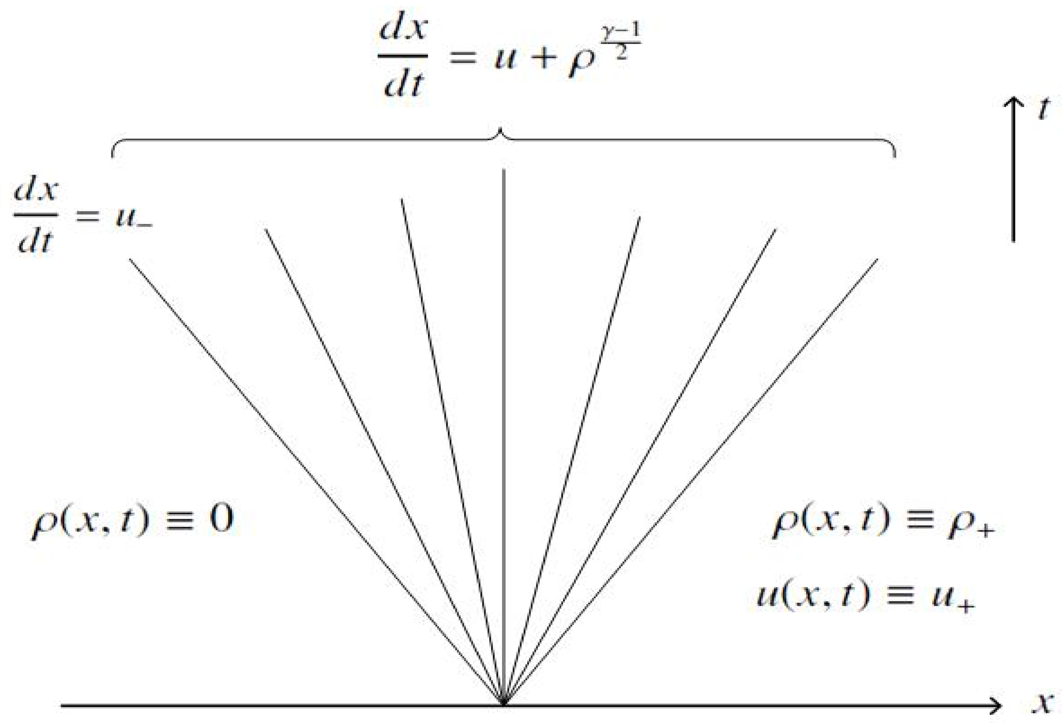
$$\begin{cases} \rho(0,x) = 0, \ x < 0, \\ (\rho,u)(0,x) = (\rho_+, u_+), \ x > 0, \end{cases} \tag{3}$$

where the left side is the vacuum state and $\rho_+ > 0, u_+$ are prescribed constants on the right state, then the Riemann problem (2), (3) admits a $2-$rarefaction wave connected to the vacuum on the left side.

2. Methods

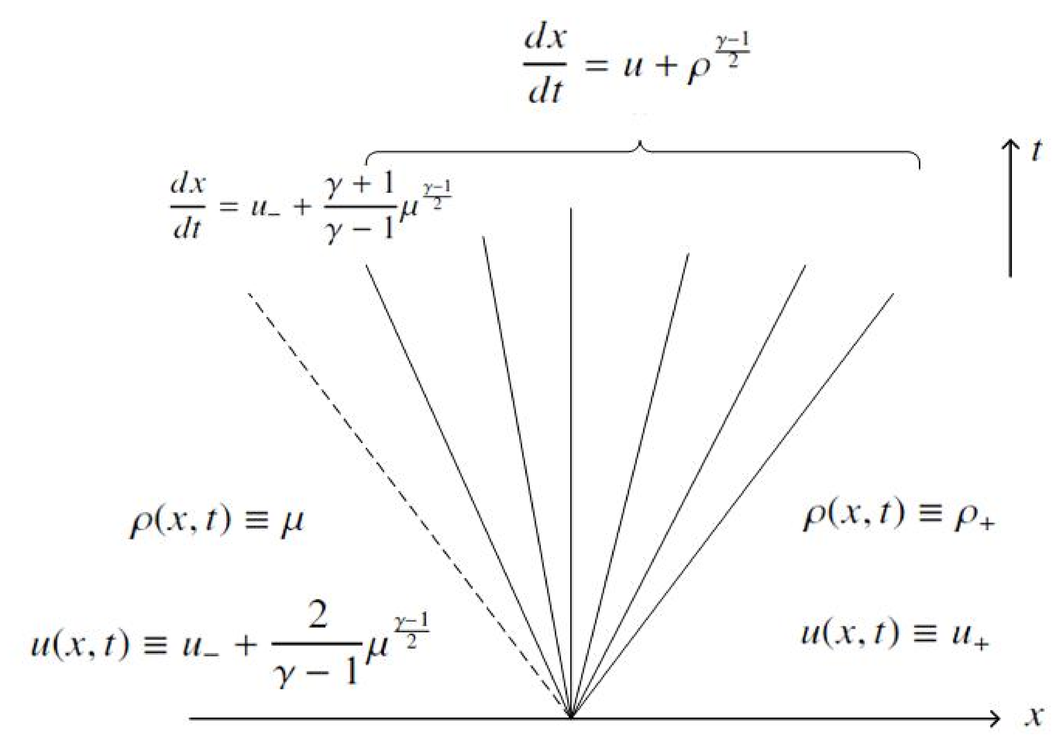
1. Lifting the vacuum region

The main novelty and difficulty of the presentation is how to control the degeneracies caused by the vacuum in the rarefaction wave.



2-rarefaction wave with vacuum

To overcome this difficulty, we first cut off the 2-rarefaction wave with vacuum along the rarefaction wave curve. More precisely, for any $\mu > 0$ to be determined, the cut-off rarefaction wave will connect the state $(\rho, u) = (\mu, u_\mu)$ and (ρ_+, u_+) where u_μ can be obtained uniquely by the definition of the 2-rarefaction wave curve.



2-rarefaction wave with lifted vacuum region

2. Construction of smooth rarefaction wave

Then an approximate rarefaction wave to this cut-off rarefaction wave will be constructed through the Burgers equation.

$$\begin{cases} w_t + w w_x = 0, \\ w(0,x) = w_\delta(x) = w(\frac{x}{\delta}) = \frac{w_+ + w_-}{2} + \frac{w_+ - w_-}{2}\tanh\frac{x}{\delta}, \end{cases} \tag{4}$$

where $\delta > 0$ is a small parameter to be determined. In fact, we choose $\delta = \epsilon^a$ in (13) with a given by (17) in the following. Note that the solution $w_\delta^r(t,x)$ of the problem (4) is given by

$$w_\delta^r(t,x) = w_\delta(x_0(t,x)), \qquad x = x_0(t,x) + w_\delta(x_0(t,x))t. \tag{5}$$

3. Energy estimation

Finally, the desired solution sequences to the compressible Navier-Stokes equations (1) could be established around the approximate rarefaction wave.
We introduce the perturbation

$$(\phi, \psi)(y,\tau) = (\rho, u)(x,t) - (\bar\rho, \bar u)(x,t), \tag{6}$$

where y, τ are the scaled variables as

$$y = \frac{x}{\epsilon}, \quad \tau = \frac{t}{\epsilon}, \tag{7}$$

and (ρ, u) is assumed to be the solution to the problem (1). Substituting (6) and (7) into (1) and using the definition for $(\bar\rho, \bar u)$, we obtain

$$\phi_\tau + \rho\psi_y + u\phi_y = -f, \tag{8}$$
$$\rho\psi_\tau + \rho u\psi_y + p'(\rho)\phi_y - \psi_{yy} = -g, \tag{9}$$
$$(\phi, \psi)(y, 0) = 0, \tag{10}$$

where

$$\begin{cases} f = \bar u_y \phi + \bar\rho_y \psi, \\ g = -\bar u_{yy} + \rho\psi\bar u_y + \bar\rho_y\Big[p'(\rho) - \frac{\rho}{\bar\rho}p'(\bar\rho)\Big]. \end{cases} \tag{11}$$

The uniform estimates to the perturbation of the solution sequences around the approximate rarefaction wave can be obtained by the following two observations. One is the fact that the viscosity ϵ can control the degeneracies caused by the vacuum in rarefaction waves by choosing suitably $\mu = \mu(\epsilon)$. In fact, we choose $\mu = \epsilon^a|\ln\epsilon|$ with a defined in (17) in the present paper. The other observation is that we can carry out the energy estimates under the a priori assumption that the perturbation is suitably small in $H^1(\mathbf{R})$ norm with some decay rate with respect to ϵ as ϵ tends to zero. The analysis is always carried out under the a priori assumptions

$$\sup_{\tau\in[0,\tau_1(\epsilon)]} \|\phi(\cdot,\tau)\|_{L^\infty} \leq \epsilon^a, \qquad \sup_{\tau\in[0,\tau_1(\epsilon)]} \|\psi_y\| \leq 1, \tag{12}$$

with a given by (17).
Note that this a priori assumption is natural but is first used in studying zero dissipation limit to our knowledge. Take

$$\mu = \epsilon^a|\ln\epsilon|, \qquad \delta = \epsilon^a, \tag{13}$$

in the sequel. Then it follows that $\mu \geq 2\epsilon^a$ if $\epsilon \ll 1$. Under the a priori assumption (12), we can get

$$\frac{\bar\rho}{2} \leq \rho \leq \frac{3\bar\rho}{2}. \tag{14}$$

With these two observations, we can close the a priori assumption and obtain the desired results.

$$\sup_{\tau\in[0,\tau_1(\epsilon)]} \int_{\mathbf{R}} \Big(\bar\rho\psi^2 + \bar\rho^{\gamma-2}\phi^2 + \phi_y^2 + \psi_y^2\Big)(\tau, y)dy + \int_0^{\tau_1(\epsilon)}\int_{\mathbf{R}} \Big[\psi_y^2 + \bar\rho^{\gamma-2}\bar u_y\phi^2 + \bar\rho\bar u_y\psi^2 + \bar\rho^{\gamma-3}\phi_y^2 + \frac{\psi_{yy}^2}{\bar\rho}\Big]dyd\tau \leq C\epsilon^{(1/2-a)}|\ln\epsilon|^{-1/2}. \tag{15}$$

3. Results

Combining all ideas and analysis together, we can get the following results:

Theorem 1 Let $(\rho^{r_2}, m^{r_2})(x/t)$ **be the 2-rarefaction wave with one-side vacuum state. Then there exists a small positive constant** ϵ_0 **such that for any** $\epsilon \in (0, \epsilon_0)$**, we can construct a global smooth solution** $(\rho^\epsilon, m^\epsilon = \rho^\epsilon u^\epsilon)(x,t)$ **with the smooth rarefaction wave initial values to the compressible Navier-Stokes equation (1) satisfying**
(1)

$$(\rho^\epsilon - \rho^{r_2}, m^\epsilon - m^{r_2}), (\rho^\epsilon, m^\epsilon)_x \in C^0((0,+\infty); L^2(\mathbf{R})),$$
$$m_{xx}^\epsilon \in L^2(0,+\infty; L^2(\mathbf{R})).$$

2) As viscosity $\epsilon \to 0$**,** $(\rho^\epsilon, m^\epsilon)(x,t)$ **converges to** $(\rho^{r_2}, m^{r_2})(x/t)$ **pointwisely except the original point** $(0,0)$**. Furthermore, for any given positive constant** h**, there exists a constant** $C_h > 0$**, independent of** ϵ**, such that**

$$\sup_{t\geq h}\|\rho^\epsilon(\cdot,t) - \rho^{r_2}(\frac{\cdot}{t})\|_{L^\infty} \leq C_h\epsilon^a|\ln\epsilon|,$$
$$\sup_{t\geq h}\|m^\epsilon(\cdot,t) - m^{r_2}(\frac{\cdot}{t})\|_{L^\infty} \leq \begin{cases} C_h\epsilon^b|\ln\epsilon|^{-\frac{1}{2}}, & \text{if} \quad 1 < \gamma < 3, \\ C_h\epsilon^{\frac{1}{\gamma+4}}|\ln\epsilon|, & \text{if} \quad \gamma \geq 3, \end{cases} \tag{16}$$

with the positive constants $a,\ b$ **given by**

$$a = \begin{cases} \frac{1}{6} & \text{if} \quad 1 < \gamma \leq 2, \\ \frac{1}{\gamma+4}, & \text{if} \quad \gamma > 2. \end{cases} \tag{17}$$

and

$$b = \begin{cases} \frac{1}{8} & \text{if} \quad 1 < \gamma \leq 2, \\ \frac{\gamma+1}{4(\gamma+4)}, & \text{if} \quad 2 < \gamma < 3. \end{cases} \tag{18}$$

4. Conclusions

Given a rarefaction wave with one-side vacuum state to the compressible Euler equations, we can construct a sequence of solutions to one-dimensional compressible isentropic Navier-Stokes equations which converge to the above rarefaction wave with vacuum as the viscosity tends to zero. Moreover, the uniform convergence rate is obtained.

References

[1] S. Bianchini and A. Bressan, *Vanishing viscosity solutions of nonlinear hyperbolic systems*, Ann. of Math. (2) **161** (2005), 223–342.

[2] G. Q. Chen and M. Perepelitsa, *Vanishing viscosity limit of the Navier-Stokes equations to the Euler equations for compressible fluid flow*, Comm. Pure Appl. Math. **63** (2010), 1469–11504.

[3] J. Goodman and Z. P. Xin, *Viscous limits for piecewise smooth solutions to systems of conservation laws*, Arch. Ration. Mech. Anal. **121** (1992), 235–265.

[4] D. Hoff and T. P. Liu, *The inviscid limit for the Navier-Stokes equations of compressible, isentropic flow with shock data*, Indiana Univ. Math. J. **38** (1989), 861–915.

[5] Q. S. Jiu, Y. Wang, and Z. P. Xin, *Vacuum behaviors around rarefaction waves to 1D compressible Navier-Stokes equations with density-dependent viscosity* (2011). preprint.

[6] T. P. Liu and J. Smoller, *On the vacuum state for the isentropic gas dynamics equations*, Adv. in Appl. Math. **1** (1980), 345–359.

[7] M. Perepelitsa, *Asymptotics toward rarefaction waves and vacuum for 1-D compressible Navier-Stokes equations*, SIAM J. Math. Anal. **42** (2010), 1404–1412.

[8] Z. P. Xin, *Zero dissipation limit to rarefaction waves for the one-dimensional Navier-Stokes equations of compressible isentropic gases*, Comm. Pure Appl. Math. **46** (1993), 621–665.

[9] S. H. Yu, *Zero-dissipation limit of solutions with shocks for systems of hyperbolic conservation laws*, Arch. Ration. Mech. Anal. **146** (1999), 275–370.