# SHOCK WAVES IN QUASI-THERMAL-INCOMPRESSIBLE MATERIALS

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**Abstract.** Incompressibility is a useful idealization for materials characterized by an extreme resistance to volume changes. For pure mechanical problems, i.e. where no change in temperature is involved, an incompressible material is easily understood as a material whose density is constant; in this case the solutions of model equations for incompressible fluids are obtainable as the limit case of the corresponding models involving compressible fluids [1].

For thermomechanical problems, i.e. when the processes are not isothermal, the definition itself of incompressibility is not straightforward and several models have been proposed. The first model of incompressibility was characterized by the independence of all the constitutive equations on the pressure [2], which leads to the conclusion - in strike contrast with experimental evidence - that the density must be constant [3]. A second, less restrictive, model requires that the only constitutive function independent of the pressure is the specific volume [4]. Such a definition of incompressibility allows to avoid the problems raised by Müller's definition but it is still not satisfactory as in this model instabilities affect wave propagation: since the chemical potential is not concave, the sound velocity might become imaginary, therefore losing the hyperbolicity of the system of Euler equations.

In order to solve these inconveniences, a new model of incompressibility has recently been proposed [5]. According to this model, a material is called *quasi-thermal-incompressible (QTI)* if the entropy principle and the thermodynamic stability condition are satisfied and the specific volume, *V*, and the specific internal energy, ε, differ to order $\delta^2$ from functions depending only on the temperature $T$:

$$V(p,T)=\hat{V}(T)+\mathcal{O}(\delta^2) \quad \text{with} \quad \hat{V}'(T)=\mathcal{O}(\delta), \qquad \varepsilon(p,T)=\hat{\varepsilon}(T)+\mathcal{O}(\delta^2)$$

where δ is a small non-dimensional parameter such that $\delta=\alpha_0 T_0$ being $\alpha_0$ the thermal expansion coefficient at the reference state. Moreover the compressibility factor $\beta_0$ at the reference state is assumed to be of order $\delta^2$. It is remarkable that QTI materials are compressible fluids that approximate incompressible fluids to order $\delta^2$ in the sense of Müller's definition.

The purpose of this work in progress is to analyze wave propagation, in particular shock waves, in QTI materials. The limit case with δ→0 (corresponding to ideal incompressible materials) is going to be investigated as well.

## I) INTRODUCTION

Although fully incompressible materials do not exist in nature, incompressibility is a useful idealization when modeling materials characterized by extreme resistance to volume changes.
From a mathematical point of view, compressible/incompressible materials have different treatments:
- for compressible materials, the pressure is a constitutive function;
- for incompressible materials, the pressure is a Lagrange multiplier (associated to the constraint of incompressibility).
In order to compare the solutions of compressible/incompressible materials, it is convenient to choose the pressure *p* (instead of the density) as a thermodynamic variable along with temperature $T$:

$V=V(p,T)$ specific volume — $p$ pressure

$\varepsilon=\varepsilon(p,T)$ specific internal energy — $T$ temperature

Two parameters are to be considered when analyzing the compressibility of materials :

$\alpha=\frac{V_T}{V}$ thermal expansion coefficient

$\beta=-\frac{V_p}{V}$ compressibility factor

Experimentally, it is seen that for materials treated as incompressible, the thermal expansion coefficient is *small* and compressibility factor is *very small*: it is frequently considered incompressibile a material such that *V=V(T).*

## III) MODELS OF INCOMPRESSIBLE MATERIALS

**1. Perfectly Incompressible Material** (Müller, [2]). According to this model, all the constitutive equations are independent of the pressure:

$$V=V(T), \quad \varepsilon=\varepsilon(T)$$

It is easily seen that: $\varepsilon=\mu-p\mu_p-T\mu_T \;\Rightarrow\; \varepsilon_p=-pV_p-TV_T \;\Rightarrow\; V_p=0$ — *the specific volume must be constant!* **(Müller paradox)**

**2. Quasi-Thermal-Incompressible Material** (Gouin, Muracchini, Ruggeri [3], see also [4]). Only the density is assumed to be indipendent of the pressure:

$$V=V(T), \quad \varepsilon=\varepsilon(p,T)$$

In this case: $V=\mu_p \;\Rightarrow\; \mu(p,T)=V(T)\,p+\mu_0(T), \quad \mu_T=V'p+\mu_0'$

$\varepsilon(p,T)=-TV'(T)\,p+e(T), \quad e(T)=\mu_0-T\mu_0'$

If $p\ll\frac{e(T)}{|V'|T}=p_{cr}$ then $\varepsilon(p,T)\approx\varepsilon(T)$

The main features of this model are:
- The Müller paradox is removed
- *Perfectly incompressible materials* are obtained as a limit of *quasi-thermal-incompressible (QTI) materials*
- The chemical potential is a linear function of the pressure: instabilities occur in wave propagation

**3. Extended-Quasi-Thermal-Incompressible Material** (Gouin, Ruggeri [5]). A compressible material satisfying the thermodynamic conditions is called an *extended-quasi-thermal-incompressible (EQTI) material* if:

$$\mathrm{V}(p,T)=\hat{V}(T)+\mathcal{O}(\delta^2) \quad \text{with} \quad \hat{V}'(T)=\mathcal{O}(\delta)$$
$$\varepsilon(p,T)=\hat{\varepsilon}(T)+\mathcal{O}(\delta^2)$$

An EQTI material is a stable compressible material that approximates a Müller incompressible material to order $\delta^2$

Assuming $\delta=\alpha_0T_0, \quad \beta_0p_0=\mathcal{O}(\delta^2) \quad (\delta\ll1)$ — $(V_0,p_0,t_0)$ reference state

It is easily obtained:

$$V(p,T)=V_0+\delta W(T)-\delta^2U(p,T) \qquad \textbf{(1)}$$
$$\mu=pV_0+\delta pW(T)+\delta^2\hat{\mu}(p,T)+\tilde{\mu}(T), \quad \hat{\mu}(p,T)=-\int U(p,T)\,dp$$
$$\varepsilon(p,T)=e(T)-\delta TW'(T)\,p+\mathcal{O}(\delta^2)$$

Thus, for values of the pressure such that: $p\ll p_{cr}(T)=\frac{1}{\delta}\frac{e(T)}{TW'(T)}$ we have: $\varepsilon(p,T)=e(T)+\mathcal{O}(\delta^2)$ **(2)**

Moreover, it is obtained: $\alpha=\delta\frac{W'(T)}{V_0}+\mathcal{O}(\delta^2), \quad W'(T_0)=\frac{V_0}{T_0}, \quad \beta=\delta^2\frac{U_p(p,T)}{V_0}, \quad \beta_{cr}=\delta^2\frac{TW'^2(T)}{V_0c_p}, \quad c_p\approx e'(T)$

And the thermodynamic stability is guaranteed if $U_p(p,T)>\frac{TW'^2(T)}{c_p}, \quad c_p\approx e'(T)>0$ **(3)**

*Conclusion*: A material characterized by equations **(1)** and **(2)**, satisfying **(3)** iapproximates a perfectly incompressible material (if the pressure is smaller than a critical value) and it is thermodynamically stable.

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## II) THERMODYNAMIC RESTRICTIONS

We consider thermodynamic conditions for compressible materials for which the specific volume is a function of the pressure and of the temperature: the entropy principle and the thermodynamic stability condition must be satisfied.

1. **Entropy Principle**

Gibbs equation $TdS=d\varepsilon+pdV$

Chemical potential $\mu=\varepsilon+pV-TS$

$$\Rightarrow\; d\mu=Vdp-SdT \;\Rightarrow\; V=\mu_p, \quad S=-\mu_T \;\text{(entropy)}, \quad \varepsilon=\mu-p\mu_p-T\mu_T$$

If the thermal equation of state is given (for example, it is obtained by experiments):

$$\mu=\int V(p,T)\,dp+\mu_o(T)$$
$$S=-\int V_T(p,T)\,dp-\mu_o'(T)$$
$$\varepsilon=e(T)+\int V(p,T)\,dp-pV-T\int V_T(p,T)\,dp$$

$$\mathrm{V}_p(p,T)\equiv\left(\tfrac{\partial V}{\partial p}\right)_T, \quad \mathrm{V}_T(p,T)\equiv\left(\tfrac{\partial V}{\partial T}\right)_p, \quad \mu_o'(T)\equiv\tfrac{d\mu}{dT}$$

with $e(T)=\mu_o(T)-T\mu_o'(T)$ or $\mu_o(T)=-T\int\frac{e(T)}{T^2}dT$

1. **Thermodynamic Stability**

Enthalpy $h=\varepsilon+pV$

Specific heat $c_p=h_T=e'(T)-T\int V_{TT}dp$

The chemical potential must be a concave function of pressure and temperature:

$$\mu_{pp}=V_p<0$$
$$\mu_{TT}=-\tfrac{c_p}{T}<0$$
$$J=\mu_{pp}\mu_{TT}-\mu_{pT}^2=-\tfrac{c_pV_p}{T}-V_T^2>0 \;\Rightarrow\; V_p<-\tfrac{TV_T^2}{c_p} \;\Rightarrow\; \beta>\tfrac{\alpha^2TV}{c_p}=\beta_{cr}$$

Thus, thermodinamic stability requires: $V_p<0, \quad c_p>0, \quad \beta>\beta_{cr} \quad \left(\beta_{cr}=\frac{\alpha^2TV}{c_p}\right)$

Adiabatic sound velocity: $c^2=\left(\frac{\partial p}{\partial\rho}\right)_S=-V^2\left(\frac{\partial p}{\partial V}\right)_S=\ldots=-\frac{\mu_p^2\mu_{TT}}{J} \;\Rightarrow\; c=\sqrt{\frac{V}{\beta-\beta_{cr}}}$

## IV) SHOCK WAVE PROPAGATION IN EQTI MATERIALS

An *extended-quasi-thermal-incompressible (EQTI) material* is described by the following equation of state, obtained as a linear expansion of $V$ near the reference state:

$$\boxed{V=V_0\left(1+\alpha_0(T-T_0)-\beta_0(p-p_0)\right)} \;\Rightarrow\; V=V_0\left(1+\delta\left(\frac{T-T_0}{T_0}\right)-\delta^2r\left(\frac{p-p_0}{p_0}\right)\right) \qquad \alpha_0=\delta/T_0, \quad \beta_0=\delta^2r/p_0, \quad \delta\ll1$$

$$e=c_pT$$

The constitutive functions are: $\mathrm{W}(T)=\frac{V_0}{T_0}(T-T_0), \quad U(p,T)=r\frac{V_0}{p_0}(p-p_0), \;\Rightarrow\; \varepsilon=c_pT+\frac{1}{2}pV_0\left(-2\delta\frac{T}{T_0}+\delta^2r\frac{p}{p_0}\right)$

The material is EQTI if: $\delta\ll1, \quad \beta>\beta_{cr} \;\Rightarrow\; r>r_{cr}=\frac{V_0p_0}{c_pT_0}, \quad p\ll p_{cr}=\frac{1}{\delta}\frac{c_pT_0}{V_0}$

In order to study shock wave propagation, the Rankine-Hugoniot condition is analyzed for the system of 1-D Euler equations:

Euler equation $\partial_t\mathbf{u}+\partial_x\mathbf{F}(\mathbf{u})=0, \quad \mathbf{u}\equiv\begin{pmatrix}\rho\\\rho v\\E\end{pmatrix}, \quad \mathbf{F}\equiv\begin{pmatrix}\rho v\\\rho v^2+p\\(E+p)v\end{pmatrix}$

$E=\frac{1}{2}\rho v^2+\rho\varepsilon$ total energy

$(\rho_0,v_0=0,p_0,T_0)$ unperturbed (reference) state

$(\rho,v,p,T)$ perturbed state

R-H condition

$$-s[\![\mathbf{u}]\!]+[\![\mathbf{F}]\!]=0\Rightarrow\begin{cases}-s(\rho-\rho_0)+\rho v=0\\-s\rho v+\rho v^2+p-p_0=0\\-s\left(\frac{1}{2}\rho v^2+\rho\varepsilon-\rho_0\varepsilon_0\right)+\left(\frac{1}{2}\rho v^2+\rho\varepsilon\right)v=0\end{cases}\Rightarrow\begin{cases}v=\delta(p-p_0)\sqrt{\dfrac{\delta r(p-p_0)+2p_0-2c_pr\rho_0T_0}{\rho_0p_0(\delta(p-p_0)-2c_p\rho_0T_0)}}\\T=T_0\left(1+\dfrac{2\delta(p-p_0)}{2c_p\rho_0T_0-\delta(p-p_0)}\right)\\M=\sqrt{1+\dfrac{p_0\delta(p-p_0)}{c_p\rho_0T_0(2p_0+\delta r(p-p_0)-2c_pr\rho_0T_0)}}\end{cases}$$

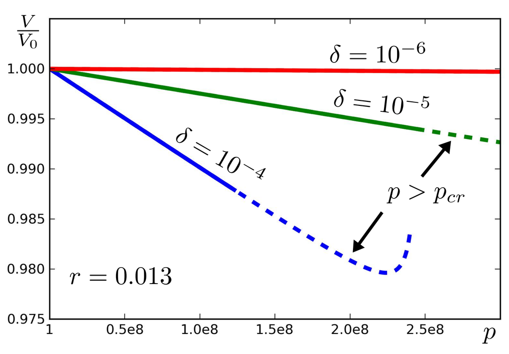
$$(s=c_0M_0+v_0=cM+v)$$

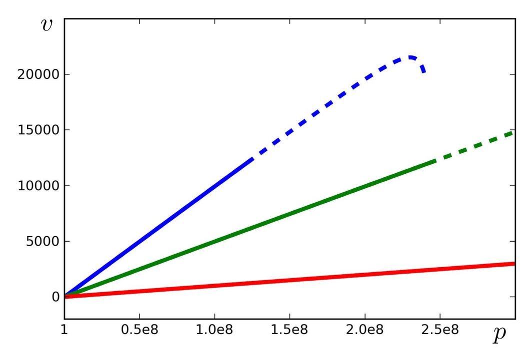
$$c=\sqrt{\frac{c_p(p_0T_0+\delta p_0(T-T_0)-\delta^2rT_0(p-p_0))}{\delta^2p_0(rcp\rho_0T_0-p_0T)}}$$

- **Material: Water**

$V_0=10^{-3}\,m^3Kg^{-1}$
$p_0=10^5\,Pa \qquad T_0=293\,K$
$\alpha_0\approx2.07\times10^{-4}K^{-1}$
$\beta_0\approx4.98\times10^{-10}Pa^{-1}$
$\Rightarrow\;\delta\approx0.061, \quad r\approx0.013$
$(r_{cr}\approx0.8\times10^{-4})$
$(p_{cr}\approx2\times10^{10}\,Pa)$

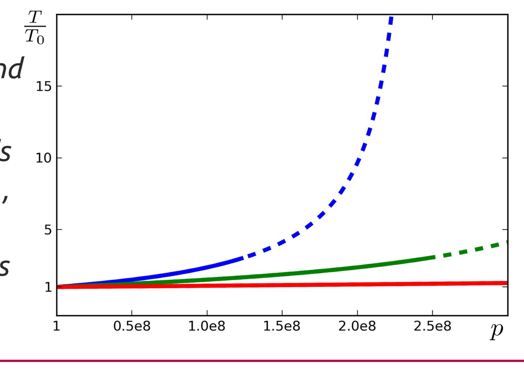
*For pressure below the critical value, water may be treated as a EQTI material*

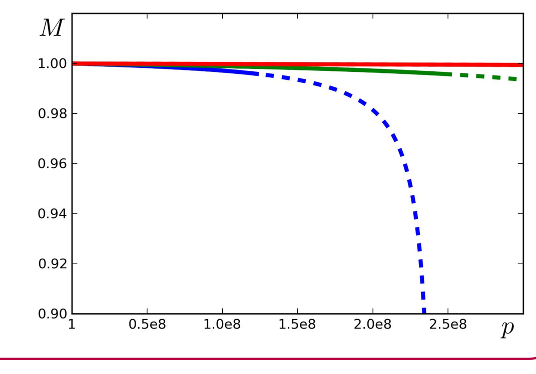



- **Analysis δ → 0**

*As the material becomes less and less compressible (δ → 0), the velocity of the shock front tends to infinity (s = c M + v ; c, v → ∞ , M → 1) and jump of density, temperature and velocity across the shock vanish.*

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