An entropy-satisfying splitting method for the Baer-Nunziato model

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Motivation

We propose a fractional step method for computing approximate solutions of the Baer-Nunziato two-phase flow model. The scheme relies on an operator splitting method corresponding to a separate treatment of fast propagation phenomena due to the acoustic waves on the one hand and slow propagation phonomena due to the fluid motion on the other. Such an approach is appropriate for designing large time-step schemes.

The isentropic Baer-Nunziato two-phase flow model

$$(\mathcal{BN}) \qquad \begin{aligned} &\partial_t\alpha_k + V_I\partial_x\alpha_k = 0,\\ &\partial_t(\alpha_k\rho_k) + \partial_x(\alpha_k\rho_k u_k) = 0,\\ &\partial_t(\alpha_k\rho_k u_k) + \partial_x(\alpha_k\rho_k u_k^2 + \alpha_k p_k(\rho_k)) - P_I\partial_x\alpha_k = 0. \end{aligned}$$

- (α_k, ρ_k, u_k): statistical fraction, density and velocity of phase k (with $\alpha_1 + \alpha_2 = 1$),
- (V_I, P_I): interfacial velocity and pressure,
- $\rho_k \mapsto p_k(\rho_k)$: barotropic pressure law.

Closure laws for (V_I, P_I):

$$(\mathcal{CL}) \qquad \begin{aligned} V_I &= (1-\mu)u_1 + \mu u_2,\\ P_I &= \mu p_1 + (1-\mu)p_2, \end{aligned}$$

where $\mu \in (0,1)$ is one of the following choices:

$$\mu \in \left\{0, 1, \frac{\alpha_1\rho_1}{\alpha_1\rho_1 + \alpha_2\rho_2}, \frac{\alpha_2\rho_2}{\alpha_1\rho_1 + \alpha_2\rho_2}\right\}.$$

Hyperbolicity:

System $(\mathcal{BN})$ admits five real eigenvalues:

$$u_k - c_k(\rho_k), \quad V_I, \quad u_k + c_k(\rho_k),$$

where $c_k(\rho_k) = \sqrt{p_k'(\rho_k)}$ are the phasic speeds of sound. The eigenvalues $u_k \pm c_k(\rho_k)$ are GNL and under the closure laws $(\mathcal{CL})$, V_I is LD. The system is hyperbolic *iff* $|u_k - V_I| \neq c_k(\rho_k)$.

Entropy inequality:

Assuming the closure laws $(\mathcal{CL})$, the entropy weak solutions of $(\mathcal{BN})$ satisfy the following inequality in the weak sense:

$$\partial_t\eta + \partial_x f_\eta \leqslant 0,$$

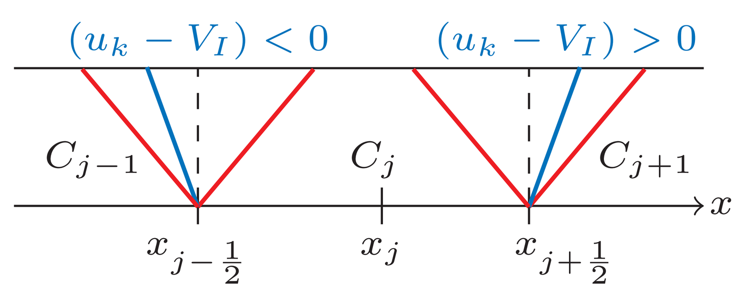
where $\eta = \sum_{k=1}^2 \alpha_k\rho_k E_k$ is the total mixture energy and $\mathbf{f}_\eta = \sum_{k=1}^2 \alpha_k(\rho_k E_k + p_k)u_k$. $E_k = u_k^2/2 + e_k$.

Time-splitting of the system

Step 1: Acoustic effects (with pressure relaxation):

$$\begin{aligned} &\partial_t\alpha_k = 0,\\ &\partial_t\alpha_k\rho_k = 0,\\ &(\alpha_k\rho_k)^0\partial_t u_k + \partial_x\alpha_k\pi_k - \Pi_I\partial_x\alpha_k = 0,\\ &(\alpha_k\rho_k)^0\partial_t\pi_k + a_k^2\partial_x\alpha_k u_k - a_k^2 V_I\partial_x\alpha_k = 0. \end{aligned}$$

This system is hyperbolic with five real eigenvalues $-a_k\tau_k$, 0 and $a_k\tau_k$, where $\tau_k = \rho_k^{-1}$. The associated Riemann problem has a unique explicit solution for all $\mu \in (0,1)$.



At $x_{j+\frac{1}{2}}$, $(u_k - V_I)$ has constant sign. The non-conservative product is distributed to the cells C_j or C_{j+1} according to this sign.
$\Longrightarrow$ Consitent definition of $(u_k)_{j+\frac{1}{2}}$ and $(V_I)_{j+\frac{1}{2}}$.

Step 2: Fluid motion in the frame of V_I:

$$\begin{aligned} &\partial_t\alpha_k = 0,\\ &\partial_t\alpha_k\rho_k + \partial_x\alpha_k\rho_k(u_k - V_I) = 0,\\ &\partial_t\alpha_k\rho_k u_k + \partial_x\alpha_k\rho_k u_k(u_k - V_I) = 0,\\ &\partial_t\alpha_k\rho_k\pi_k + \partial_x\alpha_k\rho_k\pi_k(u_k - V_I) = 0. \end{aligned}$$

$(u_k - V_I)$ is seen as a velocity field given by the first step: $(u_k - V_I)(x) := (u_k - V_I)_{j+\frac{1}{2}}$ for $x \in (x_j, x_{j+1})$.
Denoting $y_k = (u_k, \pi_k)$:

Step 2.1
$(\alpha_k\rho_k)\partial_t\tau_k = \partial_x(\alpha_k u_k) - V_I\partial_x\alpha_k = 0,$
$\partial_t y_k = 0.$

Step 2.2
$\partial_t\rho_k + (u_k - V_I)\partial_x\rho_k - \rho_k\partial_x V_I = 0,$
$\partial_t y_k + (u_k - V_I)\partial_x y_k = 0.$

Step 2.3
Pressure equilibrium: $\pi_k := p_k$.

Step 3: Transport of α_k and convection by V_I:

$$\begin{aligned} &\partial_t\alpha_k + V_I\partial_x\alpha_k = 0,\\ &\partial_t\alpha_k\rho_k + \partial_x\alpha_k\rho_k V_I = 0,\\ &\partial_t\alpha_k\rho_k u_k + \partial_x\alpha_k\rho_k u_k V_I = 0. \end{aligned}$$

V_I is seen as a velocity field given by the first step: $V_I(x) := (V_I)_{j+\frac{1}{2}}$ for $x \in (x_j, x_{j+1})$.

Step 3.1
$\partial_t\alpha_k = 0,$
$\partial_t\alpha_k\rho_k = -\alpha_k\rho_k\partial_x V_I,$
$\partial_t\alpha_k\rho_k = 0.$

Step 3.2
$\partial_t\alpha_k + V_I\partial_x\alpha_k = 0,$
$\partial_t\alpha_k\rho_k + V_I\partial_x\alpha_k\rho_k = 0,$
$\partial_t\alpha_k\rho_k u_k + V_I\partial_x\alpha_k\rho_k u_k = 0.$

Properties of the scheme

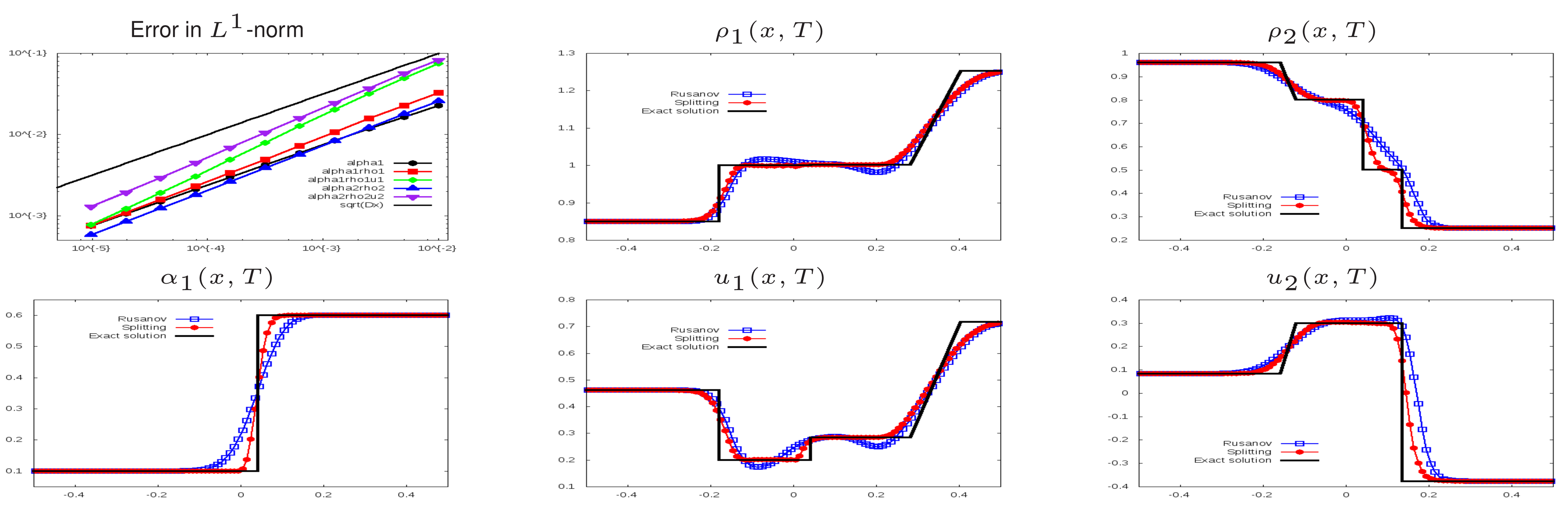
Theorem (Maximum principle): *Under some natural CFL restriction, the splitting method preserves positive values of the statistical fractions and densities.*

Theorem (Discrete entropy inequality): *There exists (a_1^{min}, a_2^{min}) such that if $a_1 > a_1^{min}$ and $a_2 > a_2^{min}$, then:*

$$\eta(\mathbb{U}_j^{n+1}) - \eta(\mathbb{U}_j^n) + \frac{\Delta t}{\Delta x}(F_\eta(\mathbb{U}_j^n, \mathbb{U}_{j+1}^n) - F_\eta(\mathbb{U}_{j-1}^n, \mathbb{U}_j^n)) \leqslant 0.$$

Numerical results

The relaxation scheme is compared with Rusanov's scheme on a Riemann problem:



Complete model with energies

The scheme can be extended to the complete model with energies:

$$\begin{aligned} &\partial_t\alpha_k + V_I\partial_x\alpha_k = 0,\\ &\partial_t(\alpha_k\rho_k) + \partial_x(\alpha_k\rho_k u_k) = 0,\\ &\partial_t(\alpha_k\rho_k u_k) + \partial_x(\alpha_k\rho_k u_k^2 + \alpha_k p_k) - P_I\partial_x\alpha_k = 0,\\ &\partial_t(\alpha_k\rho_k E_k) + \partial_x(\alpha_k(\rho_k E_k + p_k)u_k) - P_I V_I\partial_x\alpha_k = 0, \end{aligned}$$

by means of the energy/entropy duality. The physical entropies are convected phase by phase:

$$\partial_t(\alpha_k\rho_k s_k) + \partial_x(\alpha_k\rho_k s_k u_k) = 0.$$

Acknowledgements

Khaled Saleh receives a financial support by ANRT through an EDF-CIFRE contract 529/2009.

References

F. Coquel, J-M. Hérard and K. Saleh. A Splitting Method for the Isentropic Baer-Nunziato Two-Phase Flow Model. *To appear in Esaim Proceedings.*