HYPERBOLIC MODELS ARISING IN THE THEORY OF LONGITUDINAL VIBRATIONS OF ELASTIC BARS

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ABSTRACT

Longitudinal vibration of bars in mathematical physics are normally considered in terms of the classical model described by the wave equation under assumptions that the bar is thin and relatively long. More general theories were formulated, taking the effect of lateral motion of a relatively thick bar (beam) into consideration. Mathematical formulation of these models includes higher order derivatives in the equation of their motion. Rayleigh did the simplest generalization of the classical model in 1894, by including the effects of lateral motion and neglecting the shear stress. Bishop obtained the next generalization of the theory in 1952. The Rayleigh-Bishop model is described by a fourth order partial differential equation not containing the fourth time derivative. He takes into account the effects of shear stress. Both Rayleigh's and Bishop's theories consider lateral displacement being proportional to the longitudinal strain. Bishop's model was generalized by Mindlin and Hermann. They considered the lateral displacement proportional to an independent function of time and longitudinal coordinate. This result is formulated as a system of two differential equations of the second order, which could be replaced by a single equation of the fourth order, resolved with respect to the highest order time derivative. To obtain a more general class of equations, the longitudinal and lateral displacements are expressed in the form of a power series expansion in the lateral coordinate. We consider all of the above mentioned equations within a framework of a general theory of hyperbolic equations, with the goal of finding out which kind of hyperbolic equations belongs to which. We discuss the solvability of the corresponding problems.

I. INTRODUCTION

The study of general hyperbolic equations was launched by Petrovsky in his paper [1], on the Cauchy problem, where he gave a general definition of hyperbolicity. Petrovsky's initial results were complete. Further development of the theory was concerned not with obtaining new and profound results but rather with the improvement of the methods of proof and applying modern tools such as distribution theory. The Monograph of Leray [2] can be considered as the next step in this direction. Further substantial progress was made by Gårding in [3].

In 1938 Petrovsky extended his theory to general systems of partial differential equations not resolved with respect to the highest time derivatives [1]. Interest in such problems returned after Sobolev's paper [4], which was devoted to small oscillations of rotating fluid. Following Sobolev's investigations, Galpern [5] considered differerential operators of the form.

$$\sum_{j=0}^{n-m} P_{m+j}\left(D_x\right)D_t^{n-m-j} \quad (m\geq 1) \qquad \text{where } D_x=\frac{\partial}{i\partial x} \text{ and } D_t=\frac{\partial}{i\partial t} \tag{1}$$

The detailed survey of such problems can be found in the monograph of Demidenko and Uspensky [6]. In what follows, the approach on the theory of hyperbolic equations developed by Volevich is used. Some important parts of Volevich's research were devoted to the subsequent development of the works mentioned above [7]. The next fruitful step of Volevich's investigation began in 1966 in collaboration with other researchers, and in particular Gindikin. Their research brought a new approach to single out concrete classes of differential operators to which the previous results can be applied [8,9], Volevich and Gindikin obtained some outstanding results concerning the mixes problem for general hyperbolic equations and hyperbolic systems, which are summarised in [9]. A decade after these studies, many scholars are trying to extend these results or to refine them. This presentation is concerned with the latest findings of Fedotov and Volevich [10] which should provide a thorough understanding of the hyperbolic and pseudohyperbolic operators arising in the theory of longitudinal vibrations of elastic bars (beam). Pseudohyperbolic equations are studied in [11].

II. MODEL OF LONGITUDINAL VIBRATIONS

Let us consider a partial differential operator

$$Au=\alpha\left(\beta_1\frac{\partial^2}{\partial t^2}-a^2\frac{\partial^2}{\partial x^2}\right)\left(\beta_2\frac{\partial^2}{\partial t^2}-b^2\frac{\partial^2}{\partial x^2}\right)u+q^2\left(\frac{\partial^2}{\partial t^2}-c^2\frac{\partial^2}{\partial x^2}\right)u=0 \tag{2}$$

1. If $\alpha=0$ then (1) is the wave equation

$$\left(\frac{\partial^2}{\partial t^2}-c^2\frac{\partial^2}{\partial x^2}\right)u=0 \tag{3}$$

2. If $\beta_1=1,\ \beta_2=0,\ a=0,\ \alpha=\frac{p^2q^2}{b^2}$ and $p^2=\frac{\eta^2 I}{S}$ then (2) is the Rayleigh-Love equation

$$p^2\frac{\partial^4}{\partial t^2\partial x^2}u+\left(\frac{\partial^2}{\partial t^2}-c^2\frac{\partial^2}{\partial x^2}\right)=0 \tag{4}$$

3. If $\beta_1=1,\ \beta_2=0,\ a=\frac{1}{2(1+\eta)}\frac{E}{\rho},\ \alpha=\frac{p^2q^2}{b^2}$ and $p^2=\frac{\eta^2 I}{s}$ then (2) is Rayleigh-Bishop equation

$$p^2\left(\frac{\partial^4}{\partial t^2\partial x^2}+a^2\frac{\partial^4}{\partial x^4}\right)u+\left(\frac{\partial^2}{\partial t^2}-c^2\frac{\partial^2}{\partial x^2}\right)=0 \tag{5}$$

4. If $\alpha=1,\ \beta_1=1,\ a^2=\frac{1}{2(1+\eta)}\frac{E}{\rho},\ b^2=\frac{(1-\eta)}{(1+\eta)(1-2\eta)}\frac{E}{\rho},\ c^2=\frac{E}{\rho}$ and $q^2=\frac{2}{(1+\eta)(1-2\eta)}\frac{E}{\rho}$ then (2) is Mindlin-Herrmann equation

$$\left(\beta_1\frac{\partial^2}{\partial t^2}-a^2\frac{\partial^2}{\partial x^2}\right)\left(\beta_2\frac{\partial^2}{\partial t^2}-b^2\frac{\partial^2}{\partial x^2}\right)u+q^2\left(\frac{\partial^2}{\partial t^2}-c^2\frac{\partial^2}{\partial x^2}\right)u=0 \tag{6}$$

5. If $\alpha=1,\ \beta_1=0,\ \beta_2=0,\ a^2=1,\ b^2=I,\ c^2=0$ and $q^2=\frac{ES}{\rho}$ then (2) is Euler-Bernoulli beam equation

$$\left(I\frac{\partial^4}{\partial x^4}+\frac{ES}{\rho}\frac{\partial^2}{\partial t^2}\right)u=0 \tag{7}$$

6. If $\alpha=a^2,\ \beta_1=0,\ \beta_2=\frac{\rho}{E},\ a^2=b^2=1,\ c^2=0$ and $q^2=\frac{\rho}{E}\frac{S}{I}$ then (2) is Rayleigh beam equation

$$\left(\frac{\partial^4}{\partial x^4}-\frac{\rho}{E}\frac{\partial^4}{\partial x^2\partial t^2}+\frac{\rho}{E}\frac{S}{I}\frac{\partial^2}{\partial t^2}\right)u=0 \tag{8}$$

7. If $\alpha=a^2=b^2=1,\ \beta_1=\frac{\rho}{kG},\ \beta_2=\frac{\rho}{E}$ and $q=0$ then (2) is Timoshenko beam equation

$$\left(\frac{\partial^4}{\partial x^4}-(\beta_1-\beta_2)\frac{\partial^4}{\partial x^2\partial t^2}+\beta_1\beta_2\frac{\partial^2}{\partial t^2}\right)u=0 \tag{9}$$

8. Three-mode model of the longitudinal vibration of the bar

$$\left(\rho\frac{\partial^2}{\partial t^2}-(\lambda+2\mu)\frac{\partial^2}{\partial x^2}\right)\left(\rho\frac{\partial^2}{\partial t^2}-\mu\frac{\partial^2}{\partial x^2}\right)^2+\left(\rho\frac{\partial^2}{\partial t^2}-(\lambda+2)\mu\frac{\partial^2}{\partial x^2}\right)W_1+W_2=0 \tag{10}$$

Where $W_1=\frac{8}{R^2}\left\{\rho(\lambda+4\mu)\frac{\partial^2}{\partial t^2}-\left[(\lambda+2\ \mu)(\lambda+\ \ \mu)+\lambda^2+3^2\ \mu\right]\frac{\partial^2}{\partial x^2}\right\}$ and

$W_2=192\frac{\mu}{R^2}\left\{\rho(\lambda+\ \mu)\frac{\partial^2}{\partial t^2}-\ \mu(3\lambda+2\ \mu)\frac{\partial^2}{\partial x^2}\right\}$ in which $\mu=\frac{E}{2(1+\eta)}$ and $\lambda=\frac{E\eta}{(1-2\eta)(1+\eta)}$ are Lame's constants.

III. SPACES

$$(x,t)\in\mathbb{R}^2 \tag{11}$$

Let (ξ,τ) be the dual of (x,t) with respect to scalar product

$$\langle(x,t),(\xi,\tau)\rangle=x\xi+t\tau \tag{12}$$

Where $\tau\in\mathbb{C}$ and $\tau=\sigma+i\gamma$.

The corresponding (complex) Fourier transform is

$$\tilde{u}(\xi,\tau):=F_{(x,t)\to(\xi,\tau)}\left[u(x,t)\right]:=\frac{1}{2\pi}\iint u(x,t)e^{-ix\xi-t\tau}dxdt \tag{13}$$

We obtain by replacing τ by $\sigma+i\gamma$:

$$\tilde{u}(\xi,\tau):=\frac{1}{2\pi}\iint u(x,t)e^{\gamma t}e^{-ix\xi-t\tau}dxdt \tag{14}$$

it means that the complex Fourier Transform of u is the real one of $e^{\gamma t}u(x,t)$:

$$\tilde{u}(\xi,\tau)=F_{(x,t)\to(\xi,\sigma)}\left[e^{\gamma t}u(x,t)\right] \tag{15}$$

We denote by $H_s\left(\mathbb{R}^2\right)$ a Sobolev space of tempered distributions u such that $\tilde{u}(\xi,\sigma)$ is from L_2 and the norm

$$\|u\|_s=\left(\iint\left(1+\xi^2+\sigma^2\right)^s\tilde{u}(\xi,\sigma)d\xi d\sigma\right)^{1/2} \text{ is finite.} \tag{16}$$

Definition of $H_{s,\gamma}$

By definition $u(x,t)\in H_{s,\gamma}$ *if* $e^{\gamma t}u(x,t)\in H_s$. *Since*

$$\begin{aligned}F_{(x,t)\to(\xi,\sigma)}\left[e^{\gamma t}u(x,t)\right]&=\frac{1}{2\pi}\iint e^{\gamma t}u(x,t)e^{-ix\xi-it\sigma}dxdt\\&=\frac{1}{2\pi}\iint u(x,t)e^{-i(x\xi+t(\sigma+i\gamma))}dxdt\\&=\tilde{u}(\xi,\sigma+i\gamma)\end{aligned} \tag{17}$$

The norm in $H_{s,\gamma}$ is defined by

$$\|u\|_{s,\gamma}=\left(\iint\left(1+\xi^2+\sigma^2\right)^s\tilde{u}(\xi,\sigma+i\gamma)d\xi d\sigma\right)^{1/2} \tag{18}$$

or by its equivalent,

$$\|u\|_{s,\gamma}=\left(\iint_{\mathrm{Im}\tau=\gamma}\left(1+\xi^2+\sigma^2\right)^s\tilde{u}(\xi,\tau)d\xi d\tau\right)^{1/2} \tag{19}$$

It would be convenient to use another equivalent norm defined by means of pseudodifferential operator with the symbol

$$\mu_s^+(\xi,\tau)=\left(i\tau+\sqrt{1+|\xi|^2}\right)^s \tag{20}$$

where $\mathrm{Im}\tau=\gamma\leq 0$ in this case.

$$\|u\|_{s,\gamma}=\left\|e^{2\gamma t}\mu_s^+(D_x,D_t)u\right\|_0 \tag{21}$$

The space $H_{s,\gamma}$ *is the Hilbert space with the scalar product*

$$\begin{aligned}(u,v)_{s,\gamma}&=\left(\mu_s^+(D_x,D_t)u,\mu_s^+(D_x,D_t)v\right)\\&:=\iint e^{2\gamma t}\mu_s^+(D_x,D_t)u\overline{\mu_s^+(D_x,D_t)v}dxdt\end{aligned} \tag{22}$$

By definition $H_{s,\gamma}^+$ is the subspace of $H_{s,\gamma}$ formed by functions with support which is contained in the half-space $\mathbb{R}_+^2=\{(x,t)\in\mathbb{R}^2,\ t\geq 0\}$.

Finally, space $H_{s,m,\gamma}^+$ is the space with additional smoothness with respect to x. The norm of this space is defined as

$$\|u\|_{s,m,\gamma}^+=\left\|\left(1+|D_x|^2\right)^{m/2}u\right\|_{s,\gamma}^+ \tag{23}$$

Strictly Hyperbolic operator		Strictly **pseudo**hyperbolic	
Homogeneous case	Non-homogeneous case	Homogeneous case	Non-homogeneous case
DEFINITION 1 1. $A^{(0)}(\xi,\tau)=\tau^n+\alpha\tau^{n-1}\xi+\ldots$ *2. All roots of* $A^{(0)}$ *are real* *3. For* $\xi\neq 0$, *roots are pairwise distinct*	**DEFINITION 2** $A=A^{(0)}+A^{(1)}+\ldots,\ ordA^{(j)}\leq n,$ *for* $j=1,\ldots n$ $A^{(0)}$ *is Strictly Hyperbolic*	**DEFINITION 3** 1. $A^{(0)}=\tau^{n-m}+\alpha\tau^{n-m-1}\xi+\ldots$ *2. All roots of* $A^{(0)}$ *are real* *3. For* $\xi\neq 0$, *roots are pairwise distinct*	If A is strictly pseudohyperbolic in sense of Definition 4: $A=P_m(\xi)\tau^{n-m}+\ldots+P_n(\xi)=$ $=A^{(0)}+\ldots+A^{(m-\mu)}$ where $A^{(k)}=\sum_{j=0}^{n-m}a_k^{(m+j)}\xi^{m+j-k}\tau^{n-m-j}$ Theorem 4 *i)* $A^{(0)}$ *is strictly pseudohyperbolic* $A^{(m-\mu)}$ *is strictly pseudohyperbolic* *ii) All roots of* A *are real*
Theorem 1	Theorem 2	Theorem 3	
Definition 1 is equivalent to	*Definition 2 is equivalent to*	*Definition 3 is equivalent to*	**DEFINITION 4**
i) $\exists C_1,C_2$ *such that for* $\mathrm{Im}\tau\leq 0$	*i)* $\exists C_1,C_2,\gamma_0$ *such that for* $\mathrm{Im}\tau\leq\gamma_0$	*i)* $\exists C_1,C_2$ *such that for* $\mathrm{Im}\tau\leq 0$	A *is strictly pseudohyperbolic*
$C_1\leq\frac{-\mathrm{Im}\left[A^{(0)}(\xi,\tau)\overline{A_\tau^{(0)}(\xi,\tau)}\right]}{\lvert\gamma\rvert(\lvert\xi\rvert+\lvert\tau\rvert)^{2n-2}}\leq C_2,$	$C_1\leq\frac{-\mathrm{Im}\left[A^{(0)}(\xi,\tau)\overline{A_\tau^{(0)}(\xi,\tau)}\right]}{\lvert\gamma\rvert(\lvert\xi\rvert+\lvert\tau\rvert)^{2n-2}}\leq C_2,$	$C_1\leq\frac{-\mathrm{Im}\left[A^{(0)}(\xi,\tau)\overline{A_\tau^{(0)}(\xi,\tau)}\right]}{\lvert\gamma\rvert\lvert\xi\rvert^{2m}(\lvert\xi\rvert+\lvert\tau\rvert)^{2n-2m-2}}\leq C_2$	*if* $\exists C_1,C_2\geqslant 0$ *such that for* $\mathrm{Im}\tau\leq 0$ $C_1\leq\frac{-\mathrm{Im}\left[A^{(0)}(\xi,\tau)\overline{A_\tau^{(0)}(\xi,\tau)}\right]}{\lvert\gamma\rvert\left(\lvert\xi\rvert^{2m}+\lvert\xi\rvert^{2\mu}\right)(\lvert\xi\rvert+\lvert\tau\rvert)^{2n-2m-2}}\leq C_2,\xi\in\mathbb{R}.$
ii) $\exists C$ *such that for* $\mathrm{Im}\tau\leq 0$	*ii)* $\exists C,\gamma_0'$ *such that for* $\mathrm{Im}\tau\leq\gamma_0'$	*ii)* $\exists C$ *such that for* $\mathrm{Im}\tau\leq 0$	*iii) and* $\exists C$ *such that*
$\lvert A^{(0)}(\xi,\tau)\rvert\geq C\lvert\gamma\rvert(\lvert\xi\rvert+\lvert\tau\rvert)^{n-1}$	$\lvert A(\xi,\tau)\rvert\geq C\lvert\gamma\rvert(\lvert\xi\rvert+\lvert\tau\rvert)^{n-1}$	$\lvert A^{(0)}(\xi,\tau)\rvert\geq C\lvert\gamma\rvert\lvert\xi\rvert^m(\lvert\xi\rvert+\lvert\tau\rvert)^{n-m-1}$	$\lvert P_m(\xi)\rvert\geq C(\lvert\xi\rvert^m+\lvert\xi\rvert^\mu)$
iii) $\exists C_1,C_2$ *such that for* $\mathrm{Im}\tau\leq 0$	*iii)* $\exists C_1,C_2,\gamma_0$ *such that for* $\mathrm{Im}\tau\leq\gamma_0$	*iii)* $\exists C_1,C_2$ *such that for* $\mathrm{Im}\tau\leq 0$	*iv)* $\exists C_1,C_2$ *such that for* $\mathrm{Im}\tau\leq 0$
$C_1\leq\frac{-\mathrm{Im}\left(A^{(0)}(D_x,D_t),A_\tau^{(0)}(D_x,D_t)\right)_{s,\gamma}}{\lvert\gamma\rvert\left(\Vert u\Vert_{n-m-s-1,\gamma}\right)^2}\leq C_2$	$C_1\leq\frac{-\mathrm{Im}\left(A^{(0)}(D_x,D_t),A_\tau^{(0)}(D_x,D_t)\right)_{s,\gamma}}{\lvert\gamma\rvert\left(\Vert u\Vert_{n-m-s-1,\gamma}\right)^2}\leq C_2$	$C_1\leq\frac{-\mathrm{Im}\left(A^{(0)}(D_x,D_t),A_\tau^{(0)}(D_x,D_t)\right)_{s,\gamma}}{\lvert\gamma\rvert\left(\Vert\lvert D_x\rvert^m u\Vert_{n-m-s-1,\gamma}\right)^2}\leq C_2$	$C_1\leq\frac{-\mathrm{Im}(A(D_x,D_t),A_\tau(D_x,D_t))_{s,\gamma}}{\lvert\gamma\rvert\left(\Vert(\lvert D_x\rvert^m+\lvert D_x^\mu\rvert)u\Vert_{n-m-s-1,\gamma}\right)^2}\leq C_2$
$\forall u\in H_{\infty,\gamma}=\bigcap_s H_{s,\gamma}$			
	iv) $\exists C,\gamma_0'$ such that for $\mathrm{Im}\tau\leq\gamma_0'$	*iv)* $\exists C$ *such that for* $\mathrm{Im}\tau\leq 0\,\forall u\in H_{\infty,\gamma}$	*v)* $\exists C$ *such that for* $\mathrm{Im}\tau\leq 0\,\forall u\in H_{\infty,\gamma}$
$\lvert\gamma\rvert\Vert u\Vert_{n-s-1,\gamma}\leq C\Vert A^{(0)}(\xi,\tau)\Vert_{s,\gamma}$	$\lvert\gamma\rvert\Vert u\Vert_{n-s-1,\gamma}\leq C\Vert A^{(0)}(\xi,\tau)\Vert_{s,\gamma}$	$\lvert\gamma\rvert\Vert\lvert D_x\rvert^m u\Vert_{n-m-s-1,\gamma}\leq C\Vert A^{(0)}(\xi,\tau)\Vert_{s,\gamma}$	$\lvert\gamma\rvert\Vert(\lvert D_x\rvert^m+\lvert D_x^\mu\rvert)u\Vert_{n-m-s-1,\gamma}\leq C\Vert A(\xi,\tau)\Vert_{s,\gamma}$

IV. CAUCHY'S PROBLEM

Cauchy's problem consists in finding of function $u(x,t)$ satisfying equation

$$A(D_x,D_t)u=f(x,t),\quad (x,t)\in\mathbb{R}^2,\ t>0 \tag{24}$$

with initial conditions

$$D_t^j u(x,t)\Big|_{t=0}=\varphi_j(x),\quad (j=0,1,\ldots n),\quad x\in\mathbb{R} \tag{25}$$

Using the transformation $u=v+\sum_{j=0}^{n-1}t^j\varphi_j(x)$, the problem can be reduced to the homogenous Cauchy's problem that is to the analogous problem for v with the zero initial functions. If in (24) $\varphi_j\equiv 0$, the equation (23) will be valid for $t\in\mathbb{R}$ subject to u and f are extended by zero for $t<0$. Thus, Cauchy's problem can be reformulated as follows

$$\left.\begin{aligned}A(D_x,D_t)u&=f(x,t)\in H_{s,\gamma}^+\left(\mathbb{R}^2\right)\\u(x,t)&=f(x,t)=0\ \text{ for } t<0.\end{aligned}\right\} \tag{26}$$

Theorem 5 *The following statements are equivalent:*

1. *Operator* $A(D_x,D_t)$ *is strictly hyperbolic.*
2. $\forall M>0,\exists\gamma_0(M)$ *such that for* $|s|<M$ *and* $\forall\gamma\leq\gamma_0$, *problem (25) has an unique* $u\in H_{s+n-1,\gamma}^+\left(\mathbb{R}^2\right)$, *moreover, the following estimate is true:*

$$\|u\|_{s+n-1,\gamma}^+=\frac{c}{|\gamma|}\|f\|_{s,\gamma}^+ \tag{27}$$

where c *is independent of* γ.

Theorem 6 *If operator* $A(D_x,D_t)$

1. *is strictly pseudohyperbolic*
2. $\mu=0$ *(as in examples of II)*
3. $\left(1+x^2\right)^{1/2}f(x,t)\in H_{s,\gamma}^+,\ s\geq 0,\ l>1/2$

Then problem (26) has a unique solution $u\in H_{s+n-m,m,\gamma}^+$ *and* $\|u\|_{s+n-m-1,m,\gamma}^+=\frac{c}{|\gamma|}\|f\|_{s,l,\gamma}^+$ (28)

V. CONCLUSION

The solvability of the mixed problems for strictly hyperbolic equation (even for variable coefficients) was studied in [10]. At present they are no analogous results for pseudohyperbolic operators, except what it exposed in [11]. In spite of the absence of general theory of solvability of pseudohypyperbolic problems, in certain cases, it is possible to solve analytically a mixed problem, for example, Rayleigh-Bishop equation (even for variable coefficients) [12], [13], [14]. The idea of theorem 5 (and partly of theorem 6) is that hyperbolicity (pseudohybolicity) can be reformulated in term of Cauchy's problem.

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