Shallow water equations for horizontal-shear flows: characteristics, analytical and numerical solutions

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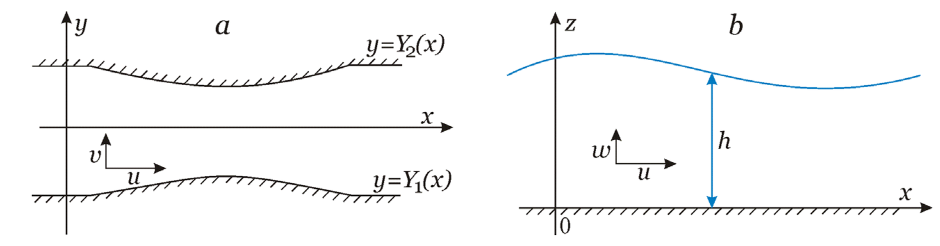
We consider the motion of an ideal fluid with free boundary $z = h(t,x,y)$ in an open channel of variable cross-section with even bottom $z = 0$ and lateral walls $y = Y_1(x)$ and $y = Y_2(x)$ in a gravity field.



The equations of motion in dimensionless variables are written as

$$u_t + (\mathbf{v}\cdot\nabla)u + p_x = 0,\quad \varepsilon^2(v_t + (\mathbf{v}\cdot\nabla)v) + p_y = 0,$$

$$\varepsilon^2(w_t + (\mathbf{v}\cdot\nabla)w) + p_z = -g,\quad \nabla\cdot\mathbf{v} = 0;\quad w = 0\quad (z = 0);$$

$$h_t + uh_x + vh_y = w,\quad p = p_0\quad (z = h);\quad uY_i'(x) = v\quad (y = Y_i).$$

Here $\mathbf{v} = (u, v, w)$ is the velocity vector; p is the pressure; g — acceleration due to gravity; $\varepsilon = l/L$ (L — the scale on the x direction, l — on the y and z).

Consider the class of fluid flows in which $u_z = 0$, $v_z = 0$. The long-wave model ($\varepsilon^2 \to 0$) becomes (Chesnokov, Liapidevskii, 2009):

$$u_t + uu_x + vu_y + gh_x = 0,\quad h_y = 0,$$

$$h_t + (uh)_x + (vh)_y = 0,\quad uY_i'(x) - v\big|_{y=Y_i} = 0. \qquad (1)$$

A consequence of Eqs. (1) is the conservation of the potential vorticity $\Omega = u_y/h$ along the trajectories: $\Omega_t + u\Omega_x + v\Omega_y = 0$.

Transform to semi-Lagrangian coordinates by the change of the variable $y = \Phi$, where $\Phi(t,x,\lambda)$ is a solution of the Cauchy problem (Zakharov, 1980)

$$\Phi_t + u(t,x,\Phi)\Phi_x = v(t,x,\Phi),\quad \Phi\big|_{t=0} = \lambda Y_2(x) + (1-\lambda)Y_1(x)$$

In the new variables we have the integrodifferential system of equations

$$u_t + uu_x + gh_x = 0,\quad H_t + (uH)_x = 0,\quad h = \frac{1}{Y}\int_0^1 H\,d\lambda \qquad (2)$$

for determining the functions $u(t,x,\lambda)$ and $H(t,x,\lambda) = h\Phi_\lambda > 0$. Here $Y(x) = Y_2(x) - Y_1(x) > 0$ is the given channel width.

Generalized characteristics

The system (2) can be presented as

$$\mathbf{U}_t + \mathbf{A}\langle\mathbf{U}_x\rangle = \mathbf{G}, \qquad (3)$$

where $\mathbf{U}(t,x,\lambda)$ — unknown vector function; $\mathbf{A} : B \to B$ — operator, acting on variable λ; and B — Banach space of the vector functions.

The characteristics of system (3) is given by the equation $x'(t) = k^\alpha(t,x)$, where k^α is the eigenvalue of the problem

$$(\mathbf{F}^\alpha, (\mathbf{A} - k^\alpha I)\langle\varphi\rangle) = 0.$$

The solution of this equation for the functional $\mathbf{F}^\alpha$ is sought in the class of generalized functions. The functional $\mathbf{F}^\alpha$ acts on the variable λ; (t and x are treated as parameters); I is the identity map; and φ is a trial function.

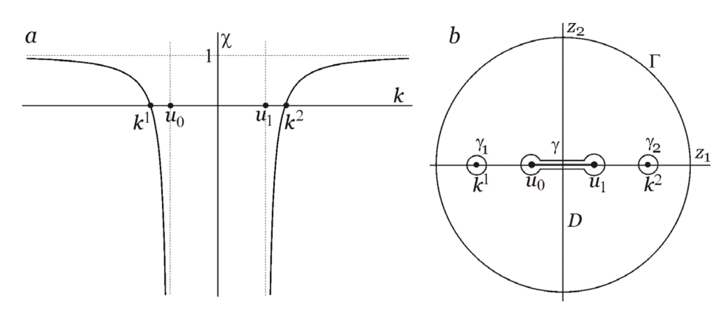
Relation on the characteristic: $(\mathbf{F}^\alpha, \mathbf{U}_t + k^\alpha\mathbf{U}_x) = (\mathbf{F}^\alpha, \mathbf{G})$.

Definition (Teshukov, 1985). System (3) is generalized hyperbolic if all eigenvalues $k^\alpha(t,x)$ are real and the set of relations on the characteristic is equivalent to (3), i. e. the eigenfunctionals $\{\mathbf{F}^\alpha\}$ represent a full basis.

The characteristic equation for the perturbation propagation velocity k in the horizontal-shear flow is

$$\chi(k) = 1 - \frac{g}{Y}\int_0^1 \frac{H\,d\lambda}{(u-k)^2} = 0.$$

We assume that $u_\lambda > 0$, $H > 0$. This equation has two real roots: $k^1 < u_0 = u(t,x,0)$ and $k^2 > u_1 = u(t,x,1)$. In addition there is continuous spectrum of characteristic velocities $k = u(t,x,\nu)$, $\nu \in [0,1]$.



A distinctive feature of integrodifferential models is the presence of both discrete and continuous spectrum of characteristic velocities.

The hyperbolicity conditions of the system (2) are formulated in terms of the analytical function $\chi(z)$, or more precisely, its limiting values on the segment $[u_0, u_1]$:

$$\chi^\pm(u) = 1 + \frac{g}{Y}\Big(\frac{1}{\Omega_1(u_1-u)} - \frac{1}{\Omega_0(u_0-u)} - \int_0^1\Big(\frac{1}{\Omega'}\Big)_\nu\frac{d\nu}{u'-u} \mp \frac{g\pi i}{u_\lambda}\Big(\frac{1}{\Omega}\Big)_\lambda\Big).$$

Here $\Omega = u_\lambda/H$ — potential vorticity; $f = f(\lambda)$, $f' = f(\nu)$.

Theorem (Chesnokov, Liapidevskii, 2011) The conditions

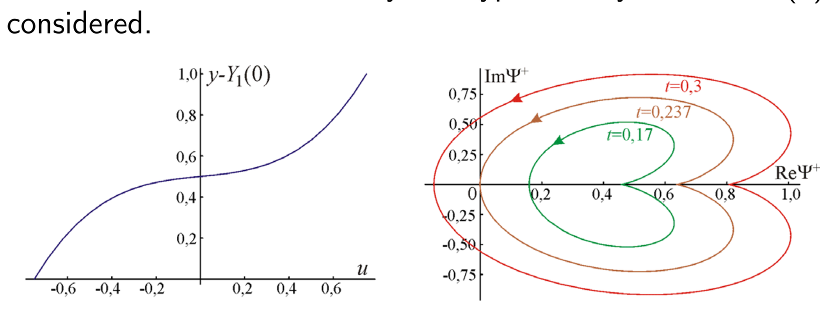
$$\chi^\pm \neq 0,\quad \Delta\arg\frac{\chi^+(u)}{\chi^-(u)} = 0 \qquad (4)$$

($\Delta\arg\chi^\pm$ is the increment of the argument of the complex function $\chi^\pm$ as λ changes from 0 to 1 for fixed t and x) are necessary and sufficient for hyperbolicity of Eqs. (2) if u and H are smooth and $u_\lambda > 0$, $H > 0$.

Validation of the hyperbolicity conditions. Consider the solution of Eqs. (2)

$$u = (x + C(\lambda))t^{-1},\quad H = t^{-1};\quad C^3 + aC - \lambda + 1/2 = 0.$$

that describes the fluid spreading under the pressure action in an open channel of constant cross-section. Let us verify the hyperbolicity conditions (4) for the solution considered.



Velocity profile u for fixed t and x; parametric representation of the real and imaginary parts of the function $\Psi^\pm$ are presented. Here $g = 1$, $Y = 1$, $\Psi^\pm = (u_1 - u)(u - u_0)\chi^\pm(u)$; $a = 5/48$, $u_1 = -u_0 = 3/4$.

This example shows that the system (2) can change its type in the process of flow evolution, which corresponds to long-wave instability for the considered velocity distribution.

Sub- and supercritical steady-state horizontal-shear flows

We take the stream function as the Lagrangian coordinate λ, so that

$$H = hy_\lambda = h/\lambda_y = 1/u.$$

To be specific, let $u > 0$. Then, as a result of integrating Eqs. (2) we obtain

$$u = \sqrt{2(C(\lambda) - gh)},\quad H = 1/\sqrt{2(C(\lambda) - gh)}$$

Here $C(\lambda) > 0$ is an arbitrary function and the fluid depth $h(x)$ can be found from the closing relation

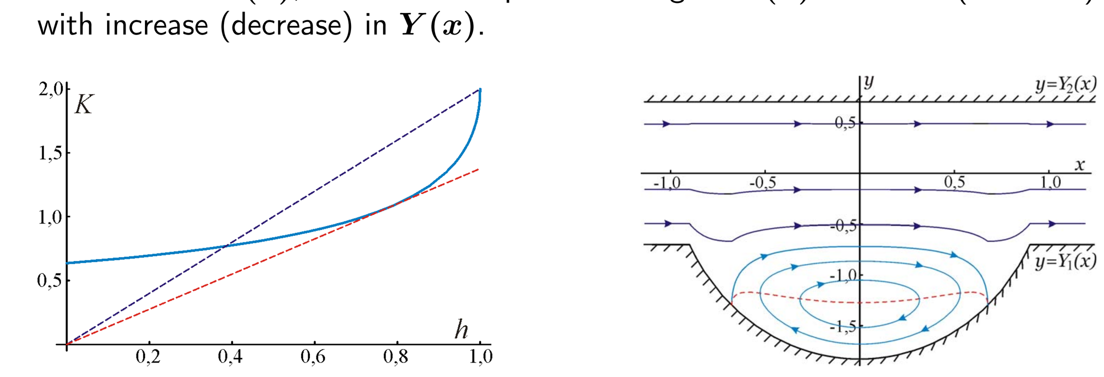
$$K(h) = Y(x)h,\quad K(h) = \int_0^1 \frac{d\lambda}{\sqrt{2(C(\lambda) - gh)}}$$

The steady-state flow for which the following inequality holds

$$S = 1 - \frac{K'(h)}{Y(x)} = 1 - \frac{g}{Y}\int_0^1 \frac{H\,d\lambda}{u^2} < 0$$

will be called **subcritical**, otherwise — **supercritical**.

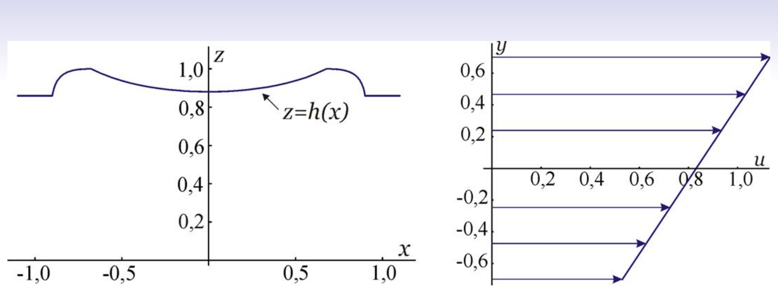
In view of $h'(x) = -(YS)^{-1}Y'(x)h$ in the **subcritical** flow regime the fluid depth $h(x)$ **increases** (decreases) with **increase** (decrease) in the channel cross-section $Y(x)$, while in the supercritical regime $h(x)$ decreases (increases) with increase (decrease) in $Y(x)$.



Exact solutions describing different flow regimes were constructed and their properties were studied (Chesnokov, Liapidevskii, 2009).

Flow with recirculation zones. Subcritical flow past a local channel expansion: the continuation of solution with the inclusion of the recirculation region

$$u = \mp\sqrt{2(G(\lambda) - gh)},\quad H = \mp 1/\sqrt{2(G(\lambda) - gh)}$$



Fluid depth, velocity profile (before entering the zone of the channel expansion) and streamlines in a subcritical steady-state flow with a recirculation zone.

Similar solutions are known for the vertical-shear flows over an uneven bottom (Varley, Blythe, 1983; Teshukov, Budlal, 2006).

Flows with a critical layer. The equations of motion (2) admit solutions describing the fluid flow with a critical layer (Chesnokov, Kovtunenko, 2011). The solution of Eq. (2) of the form

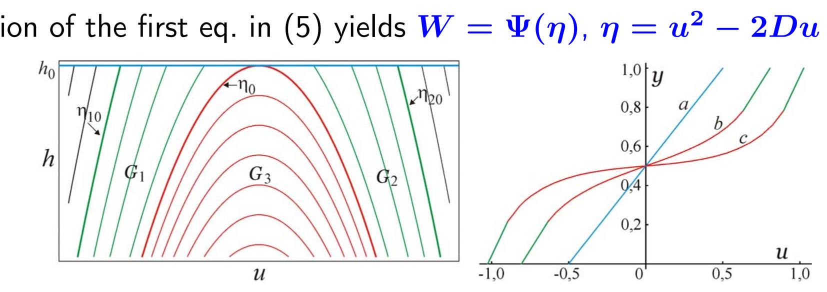
$$u = u(\zeta, y),\quad h = h(\zeta),\quad \zeta = x - Dt$$

describes a traveling wave (here we assume that $Y_i = \mathrm{const}$). For this class of solutions, system (2) (in the Vlasov-like formulation $W = \Omega^{-1}$) becomes

$$(u - D)W_\zeta - gh_\zeta W_u = 0,\quad h = \frac{1}{Y}\int_{u_0}^{u_1} W\,du \qquad (5)$$

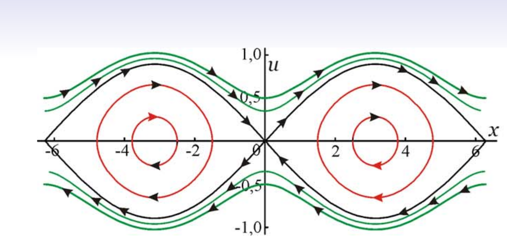
$$(u_i - D)u_i'(\zeta) + gh'(\zeta) = 0 \qquad (i = 0, 1)$$

Integration of the first eq. in (5) yields $W = \Psi(\eta)$, $\eta = u^2 - 2Du + 2gh$.



In the domains G_1 and G_2 the solution is constructed by the method of characteristics. No characteristic with origin on the curve $h = h_0$ comes to the domain G_3. To determine $W = \Psi_3(\eta)$ in this domain, we use the second eq. in (5), which can be converted to the Abel equation ($s = 2gD - D^2$):

$$\int_s^{\eta_0}\frac{\Phi_3(\eta)\,d\eta}{\sqrt{\eta - s}} = \frac{(s + D^2)Y}{2g} - \frac{1}{2}\int_{\eta_0}^{\eta_{10}}\frac{\Phi_1(\eta)\,d\eta}{\sqrt{\eta - s}} - \frac{1}{2}\int_{\eta_0}^{\eta_{20}}\frac{\Phi_2(\eta)\,d\eta}{\sqrt{\eta - s}}.$$



The trajectories have the cat-eye shape characteristic of flows with a critical layer.

The conditions of hyperbolicity (4) of the equations (2) for solutions from the class of traveling waves are valid only for a sufficiently small change in the fluid depth h.

Balance laws. In the study of shear discontinuous flows (hydraulic jump) we propose to use the balance laws ($y = \Phi(t,x,\lambda)$, $H = \Phi_\lambda$)

$$(u - u_0)_t + \Big(\frac{u^2 - u_0^2}{2}\Big)_x = 0,\quad H_t + (uH)_x = 0,$$

$$\Big(\int_0^1 uH\,d\lambda\Big)_t + \Big(\int_0^1 u^2H\,d\lambda + \frac{gYh^2}{2}\Big)_x = \frac{gY'h^2}{2}$$

which express the balance of the relative local momentum, mass and total momentum of the liquid layer. Similar relationships were used to simulate the hydraulic jump in the framework of Benney equations (vertical-shear flows) (Teshukov, Russo, Chesnokov, 2004).

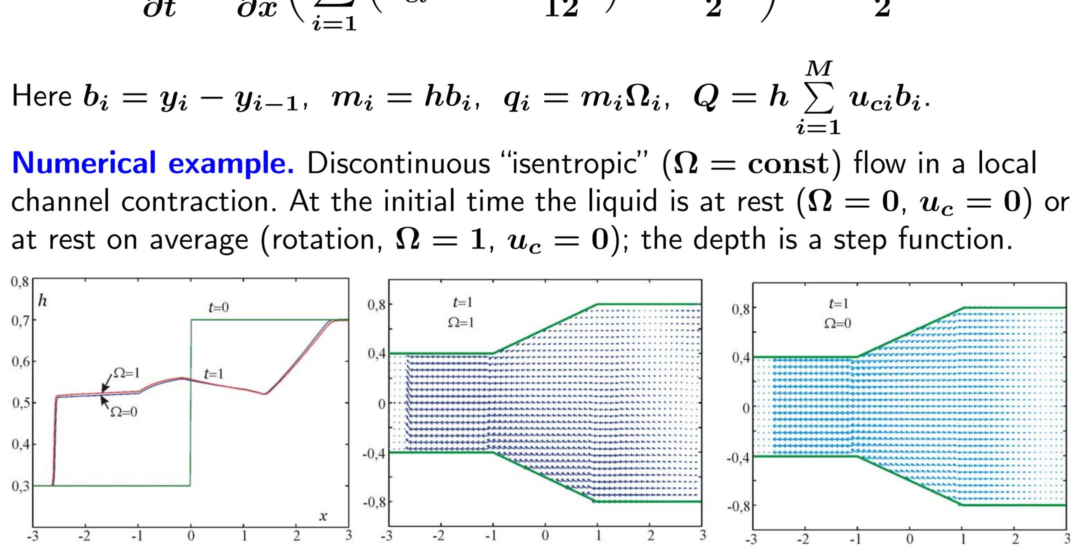
Multilayered approximation ($u = (y - y_{i-1})h\Omega_i + u_{i-1}$, $y \in [y_{i-1}, y_i]$)

$$\frac{\partial m_i}{\partial t} + \frac{\partial}{\partial x}(u_{ci}m_i) = 0,\quad \frac{\partial q_i}{\partial t} + \frac{\partial}{\partial x}(u_{ci}q_i) = 0,$$

$$\frac{\partial Q}{\partial t} + \frac{\partial}{\partial x}\Big(\sum_{i=1}^M\Big(u_{ci}^2m_i + \frac{q_i^2m_i}{12}\Big) + \frac{gYh^2}{2}\Big) = \frac{gY'h^2}{2}.$$

Here $b_i = y_i - y_{i-1}$, $m_i = hb_i$, $q_i = m_i\Omega_i$, $Q = h\sum_{i=1}^M u_{ci}b_i$.

Numerical example. Discontinuous "isentropic" ($\Omega = \mathrm{const}$) flow in a local channel contraction. At the initial time the liquid is at rest ($\Omega = 0$, $u_c = 0$) or at rest on average (rotation, $\Omega = 1$, $u_c = 0$); the depth is a step function.



Gas dynamics analogy. Let us perform the averaging equations (2) across the width of the channel under the assumption that (follows Teshukov, 2007)

$$u(t,x,y) \approx (y - Y_1(x))h(t,x)\Omega(t,x) + u_0(t,x).$$

This leads to the system of gas dynamic type

$$\rho_t + (u_c\rho)_x = 0,\quad (\rho = hY)$$

$$(u_c\rho)_t + \Big(u_c^2\rho + \frac{\Omega^2\rho^3}{12} + \frac{g\rho^2}{2Y}\Big)_x = \frac{g\rho^2Y'}{2Y^2},$$

$$\Big(u_c^2\rho + \frac{\Omega^2\rho^3}{12} + \frac{g\rho^2}{Y}\Big)_t + \Big(u_c^3\rho + \frac{u_c\Omega^2\rho^3}{4} + \frac{2gu_c\rho^2}{Y}\Big)_x = 0$$

Here $\rho = hY$ plays the role of the density of the "gas" and the entropy $s = \ln\Omega^2$ is related to the potential vorticity; u_c is the average velocity. The equations of state for the "gas" are written as ($Y = \mathrm{const}$)

$$p(\rho, s) = \frac{g\rho^2}{2Y} + \frac{\rho^3\exp(s)}{12},\quad e(\rho, s) = \frac{g\rho}{2Y} + \frac{\rho^2\exp(s)}{24}.$$

Fluid flows with $\Omega = \mathrm{const}$ correspond to isentropic gas flows, and the classical shallow-water equations are derived by passing to the limit $s \to -\infty$.

Conclusion

- Nonlinear integrodifferential models of shallow flow with continuous velocity distribution are derived and stability of shear flows in terms of hyperbolicity of the governing equations is studied.
- The concepts of sub- and supercritical flows are introduced for the model describing the steady-state shear shallow flows. Analytical solutions for flows with the formation of recirculation zones are obtained.
- The conservative form of the governing equations is proposed and gas dynamics analogy is established. These equations are used to perform numerical calculations of wave propagation.

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