Wave propagation in discrete heterogeneous media

Aurora Marica* and Enrique Zuazua

BCAM - Basque Center for Applied Mathematics, Bilbao, Basque Country, Spain

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Objectives and problem formulation

Wave equation (WE) $(x \in \mathbb{R},\ t > 0)$

$$\boxed{\partial_{tt}u - \partial_{xx}u = 0,\ u(x,0) = u^0(x),\ u_t(x,0) = u^1(x)}$$

Observability inequality (OI) $(\Omega := \mathbb{R} \setminus (-1,1),\ T \geq 2)$

$$\boxed{||(u^0,u^1)||^2_{\dot{H}^1 \times L^2(\mathbb{R})} \leq C(T) \int_0^T ||(u(\cdot,t),u_t(\cdot,t))||^2_{\dot{H}^1 \times L^2(\Omega)}\, dt}$$

Applications: control, stabilization and inverse problems

Objectives

☐ Discrete versions of OI and DE (non-uniformity in h, filtering mechanisms)
☐ Behaviour of high frequency Gaussian wave packets in complex media (complex schemes, splitting under filtering, non-uniform meshes, etc.)

Schrödinger equation (SE) $(x \in \mathbb{R}, t > 0)$

$$\boxed{i\partial_t u + \partial_{xx}u = 0,\ u(x,0) = \varphi(x)}$$

Dispersive estimates (DE): i) Gain of integrability

$$\boxed{\|u\|_{L^q_t(\mathbb{R},L^p_x(\mathbb{R}))} \leq c(p)\|\varphi\|_{L^2(\mathbb{R})}}$$

Admissibility conditions: $2 \leq p \leq \infty$ and $2/q = 1/2 - 1/p$

ii) Smoothing effect

$$\boxed{\sup_R \left(\frac{1}{R}\int_{\mathbb{R}}\int_{-R}^{R} |\partial_x^{1/2}u(x,t)|^2\, dx\, dt\right)^{1/2} \leq c\|\varphi\|_{L^2(\mathbb{R})}}$$

Applications: well-posedness of non-linear Schrödinger equation

Discontinuous Galerkin (DG) approximations

Notations: $\{\cdot\}$, $[\cdot]$ - **average/jump**; $s > 1$ - **penalty parameter**; $\mathcal{G}^h$, $\mathcal{T}^h$ - uniform grid/triangulation of size h of $\mathbb{R}$; $\mathcal{V}^h$ - space of **piecewise linear** and **discontinuous** functions

Symmetric interior penalty DG (SIPG) approximation

Bilinear form

$$\boxed{\mathcal{A}^h_s(u,v) = (\partial^h_x u, \partial^h_x v)_{L^2(\mathcal{T}^h)} - (\{\partial^h_x \mathrm{u}\}, [\mathrm{v}])_{\ell^2(\mathcal{G}^h)} - ([\mathrm{u}], \{\partial^h_x \mathrm{v}\})_{\ell^2(\mathcal{G}^h)} + \frac{s}{h}([\mathrm{u}],[\mathrm{v}])_{\ell^2(\mathcal{G}^h)}}$$

Discrete wave equation

$$\boxed{\text{Find } u^h(\cdot,t) \in \mathcal{V}^h \text{ s.t. } (u^h_{tt}(\cdot,t),\phi)_{L^2(\mathbb{R})} + \mathcal{A}^h_s(u^h(\cdot,t),\phi) = 0, \quad \forall \phi \in \mathcal{V}^h}$$

$\widehat{U}^h(\xi,t)$, the vector of **Fourier transforms** of $\{u^h(\cdot,t)\}$ and $[u^h(\cdot,t)]$, verifies the system:

$$\boxed{\widehat{U}^h_{tt}(\xi,t) + \widehat{S}^h_s(\xi)\widehat{U}^h(\xi,t) = 0, \quad \xi \in \Pi^h := [-\pi/h, \pi/h]}$$

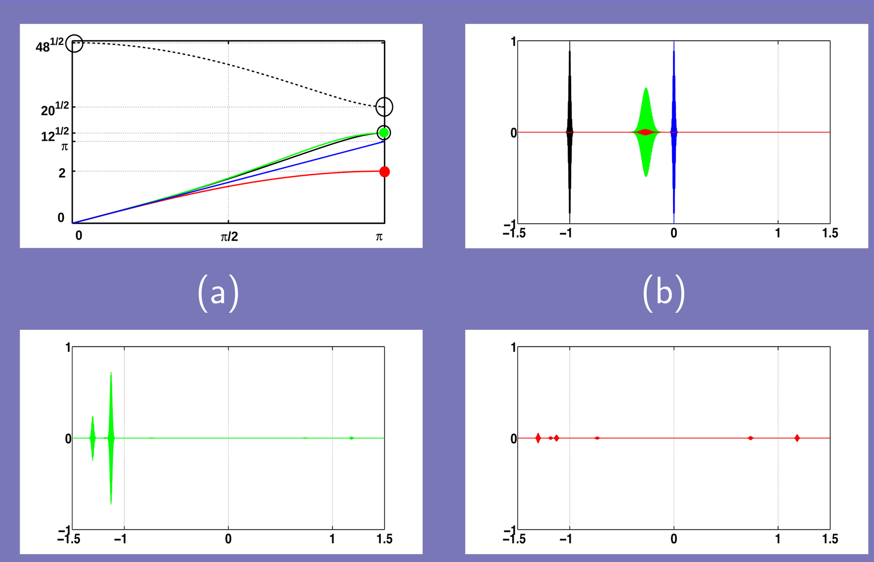
Eigenvalues of $\widehat{S}^h_s(\xi)$: **physical** and **spurious** $(\widehat{\Lambda}^h_{s,ph}(\xi)$ and $\widehat{\Lambda}^h_{s,sp}(\xi))$
Dispersion relations: $\lambda = \sqrt{\Lambda}$ (see Fig. 1(a))

Group velocities $\partial_\xi \widehat{\lambda}^h_{s,ph}$ and $\partial_\xi \widehat{\lambda}^h_{s,sp}$

They vanish for $\xi = \pi/h$, $\xi \in \{0, \pi/h\} \implies$ non-uniform OI as $h \to 0$ (see Fig. 1(b)).

Bi-grid filtering mechanism for the DG method (cf. [5])

Discrete initial data with **null jumps** + **averages obtained by a bi-grid algorithm of mesh ratio** $1/2 \Rightarrow$ uniform OI as $h \to 0$ (see Fig. 1(c-d))



(a) (b) (c) (d)

Figure 1

Legend
(a) In **black**/**dotted black**, the physical/spurious dispersion relation $\widehat{\lambda}^1_{s,ph}(\xi)$ and $\widehat{\lambda}^1_{s,sp}(\xi)$ for $s = 5$; in blue/red/green, the dispersion relations for the continuous wave equation, for its finite differences and P_1-finite element schemes (marked points = wave numbers where the **group velocity** vanish). **(b)** The average/jump components (in green/red) of the DG approximation at the wave number $\xi_0 = 49\pi/50h$ and at time $t = 1$ compared to the Gaussian initial data (in blue) and the solution of the continuous wave equation at time $t = 1$ (in black).
(c,d) The average/jump components (in green/red) of the DG approximation at the wave number $\xi_0 = 21\pi/32h$ and at time $t = 1$ under a bi-grid filtering of mesh ratio $1/2$ of the initial data.

Bi-grid algorithms

Gaussian initial data: $\varphi = \varphi^\gamma_{\xi_0}$ such that

$$\boxed{\widehat{\varphi}^\gamma_{\xi_0}(\xi) = \sqrt{\frac{2\pi}{\gamma}} \exp\left(-\frac{|\xi - \xi_0|^2}{2\gamma}\right) \chi_{\Pi^h}(\xi)}$$

$\gamma \leq h^{-2/3}$ for SE and $\gamma \leq h^{-1/2}$ for WE

Fourier representation of the **restriction operator** $\Gamma_k : \mathcal{G}^h \to \mathcal{G}^{2^k h}$

$$\boxed{\widehat{\Gamma_k f}^h(\xi) = \sum_{j=-2^{k-1}}^{2^{k-1}-1} \widehat{f}^h\left(\xi + \frac{2j\pi}{2^k h}\right)}$$

Fourier symbol of the **linear interpolation** between grids of size $2^k h$ and h

$$\boxed{\widehat{b}^h_k(\xi) := \prod_{j=1}^{k} \cos^2(2^{j-1}\xi h)}$$

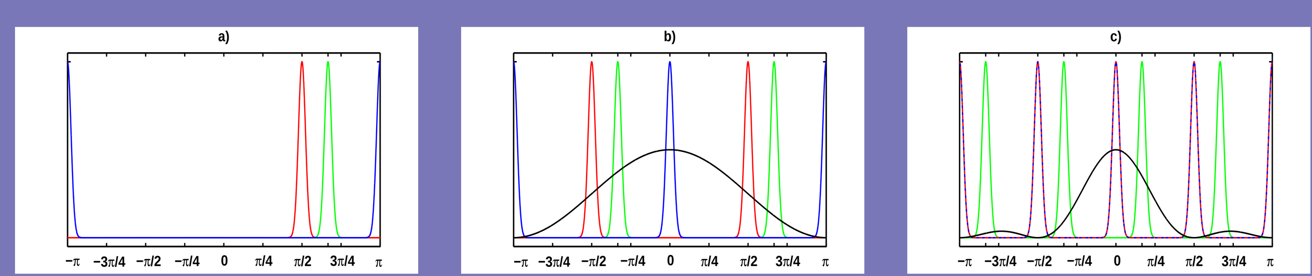


Figure 2

Legend: **a)** $\widehat{\varphi}^\gamma_{\xi_0}$ with $\xi_0 = \pi/h,\ \pi/2h,\ 2\pi/3h$ (blue, red, green) and their projections Γ_k with **b)** $k = 1$ and **c)** $k = 2$. In black, the bi-grid symbols $\widehat{b}^h_k(\xi)$

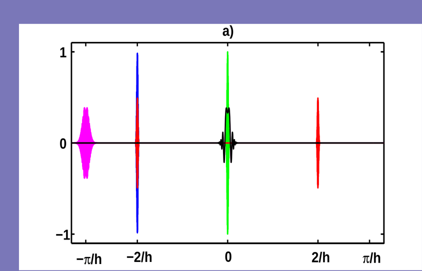


Figure 3. Solutions of continuous and finite difference discrete SE for the initial data $\varphi^\gamma_{\pi/2h}$

Legend: green - solution of continuous SE at $t = 0$, magenta - solution of continuous SE at $t = 1$, blue - solution of discrete SE without filtering at $t = 1$, and red/black - solution of discrete SE with bi-grid of ratio $1/2$ and $1/4$ at $t = 1$.

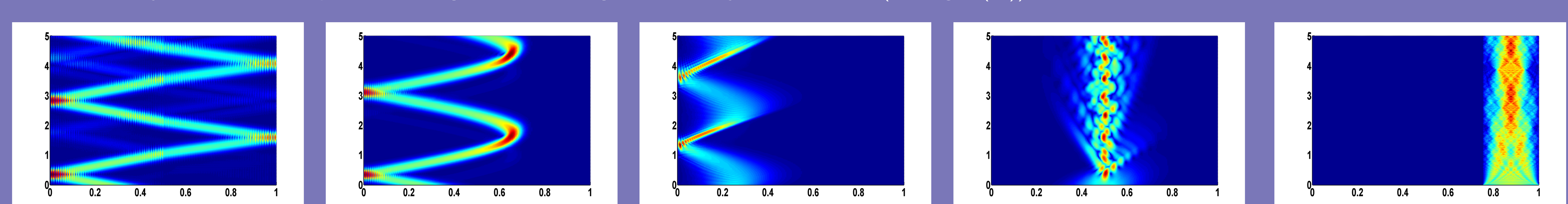
High frequency propagation of waves (WE) in discrete heterogeneous media - open problem

x=uniform grid of size h of $(0,1)$, y=non-uniform grid of $(0,1)$ and

$$\boxed{u^0(y) = \varphi^\gamma_{\xi_0}(y - y_0)\exp(i\xi_0 y_0)}$$

Legend:
(e) $h = 1/200$, uniform grid of size h for $y \in (0,1/2)$ and $h/2$ for $y \in (1/2,1)$ and $y_0 = 1/4$
(f-g) $h = 1/200$, $y = \tan(\pi x/4)$, $y_0 = 1/4$; **(h)** $h = 1/200$, $y = \sin(\pi x/3)$ for $x \in (0,1/2)$, $y = 1 - \sin(\pi(1-x)/3)$ for $x \in (1/2,1)$ and $y_0 = 1/2$; **(i)** $h = 1/100$, uniform grid of size $h/8$ and $h/4$ for $y \in (0,1/4)$ and $y \in (3/4,1)$, $y = 1/4 + \tan(x/4)/2$ for $y \in (1/4,3/4)$, and $y_0 = 7/8$

We illustrate some phenomena, most of them being pathological and requiring further analysis:
☐ reflection-transmission problem at the interface between two piecewise uniform discrete media (see Fig. 4(e))
☐ torsion of the rays of Geometric Optics, reflecting before touching the boundary of the domain (see Fig. 4(f-i))



(e) $\xi_0 = \pi/2h$ (f) $\xi_0 = \pi/2h_{min}$ (g) $\xi_0 = \pi/h_{min}$ (h) $\xi_0 = \pi/2h_{min}$ (i) $\xi_0 = \pi/2h_{min}$

Figure 4

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