A finite volume scheme for formation of SandPile based on Discontinuous flux for Conservation Laws

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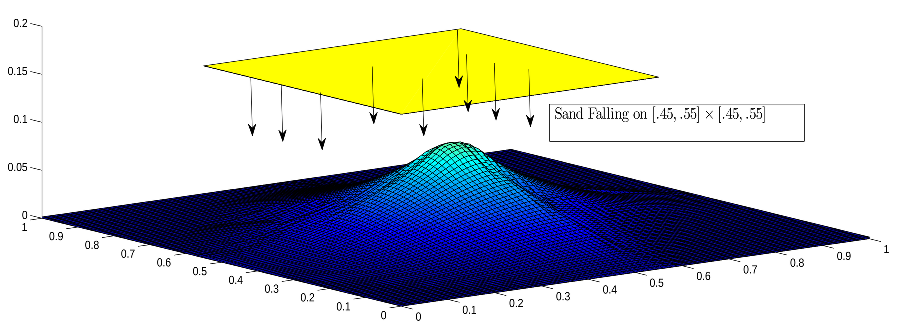
Hadeler-Kuttler Model [KH99]



$$v_t - \nabla.(v\nabla u) = -(1 - |\nabla u|)v + f, \text{ on } \Omega \times (0,T) \quad (1)$$

$$u_t = (1 - |\nabla u|)v, \text{ in } \Omega \times (0,T) \quad (2)$$

with $u_0 = 0$ on Ω and $u = 0$ on $\partial\Omega$

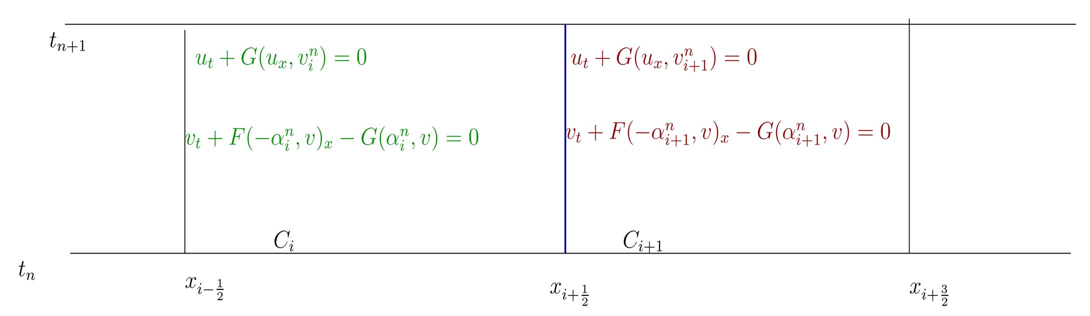
(a) Standing layer: $u(x,t)$ collects the amount of matter that remains at rest

(b)Rolling Layer: $v(x,t)$represents matter moving down along the surface of the standing layer. The stationary system **(E)** for the above system is given by:

$$\begin{array}{lll} -\nabla.(v\nabla u) = f & \text{in} & [\Omega] \\ |\nabla u| = 1 & \text{on} & \{v > 0\} \\ |\nabla u| \le 1, u, v \ge 0 & \text{in} & [\Omega] \\ u = 0 & \text{on} & \partial\Omega \end{array}$$

No Uniqueness for Stationary solution u if stationary solution $v = 0$

Techniques Used-1 dimension



Structure of the Reimann Problem for the system

Discretization of (2)

$u_t = (1 - |u_x|)\bar{v}$, $\bar{v} \approx \bar{v}_i$

- Use common indexed $\bar{v}_i$ for v. Gives Monotonicity.Extendable to 2 dimensions,Numerical Physical Estimates
- Discontinuous Hamiltonian due to v.Use Discontinuous Gudonov Flux for $|u_x|$

Discretization of (1)

$v_t - (vu_x)_x = ... + f$

- Include Source term f with $(-vu_x)_x$. Define $B_i = \int_0^{x_i} f(s)ds$ — USE [NHR09]
- Use Discontinuous Flux for discontinuous cofficient $-u_x$ — USE [Adi04]

Use [Adi04] for Discontinuous Flux idea and [NHR09]for source term.

$$u_i^{n+1} = u_i^n - \Delta t \bar{v}_i^n(-1 + \text{Gudonov Flux for}|w|)_i^n) = u_i^n - \Delta t G_i^n \quad (3)$$

$B_i \le B_{i+1}, F_1(v) = -\alpha_i^n v - B_i, F_2(v) = -\alpha_{i+1}^n v - B_{i+1}$.

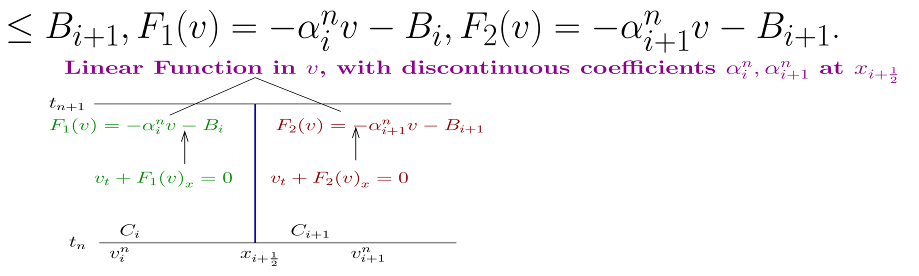
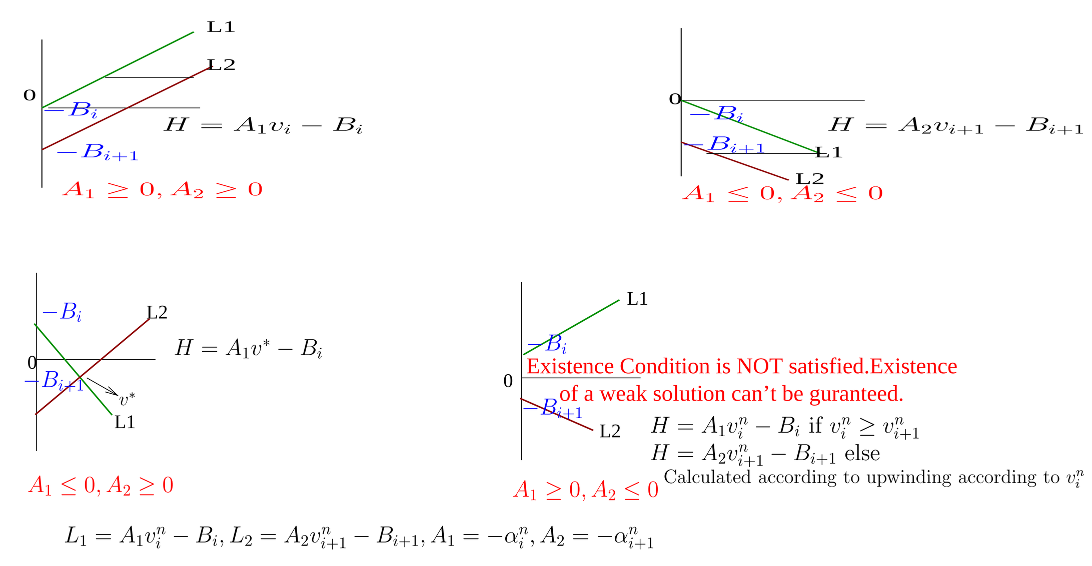


FIGURE 1: Reimann Problem for v at $x_{i+\frac{1}{2}}$

Flux Selection $H_{i+\frac{1}{2}}$ of (1)



$$v_i^{n+1} = v_i^n - \frac{\Delta t}{h}(H_{i+\frac{1}{2}}^n - H_{i-\frac{1}{2}}^n) + \Delta t v_i^n(|\alpha_i^n| - 1) \quad (4)$$

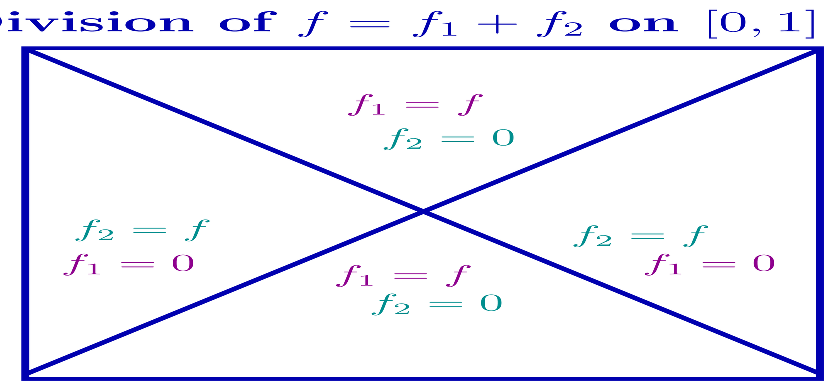
Techniques Used- dimension 2

$v_t - (vu_x)_x - (vu_y)_y = ... + f_1 + f_2$

- Use Discontinuous Flux for discontinuous cofficient $-u_x$
- Define $B_{i,k}^y = \int_0^{x_i} f_1(x, y_{k+\frac{1}{2}})dx$. Include Source term f_1 with $(-vu_x)_x$.
- Use Discontinuous Flux for discontinuous cofficient $-u_y$
- Define $B_{i,k}^x = \int_0^{y_k} f_2(x_{i+\frac{1}{2}}, y)dy$. Include Source term f_2 with $(-vu_y)_y$.

- Calculate H^x and H^y as in 1 dimension by solving Reimann Problem for v in each direction .
- $B_{i,k}^x = B^i$ for a fixed k, $B_{i,k}^y = B^k$ for a fixed i

Division of $f = f_1 + f_2$ on $[0,1] \times [0,1]$



- Technique for Equation 3

$\bar{v}$ in x-direction → $\bar{v}^x$; $\bar{v}$ in y-direction → $\bar{v}^y$; → $\bar{v}$

$$u_t = (1 - \sqrt{(u_x)^2 + (u_y)^2}\,v$$

- Gudonov Flux for $(u_x)^2$
- Gudonov Flux for $(u_y)^2$

Stability Results

- (positivity and monotonicity in u): $u^{n+1} \ge u^n \ge 0$
- (positivity in v):$v^n \ge 0$
- (gradient constraint in u):$\sup_{i,k} A_{i+j,k+j_1}^n \le 1 \forall j, j_1 \in Z^- U Z^+$,where j, j_1 depend on signs on indices of α and β,where $u^{n+1} = u^n - \Delta t * (1 - \overline{v}^{n+1}(-1 + A^n)$ under theCFL condition

$$\lambda \le \min\left(\frac{1 + \Delta t(||Du_{i,k}^n|| - 1)}{|\alpha_{i,k}^n| + |\beta_{i,k}^n|}, \frac{1}{C}\right)$$

where $C = \sup_{i,k} \frac{\overline{v}_{i,k}^{n+1}(|\alpha_{i,k}^n| + |\beta_{i,k}^n|)}{||Du_{i,k}^n||}, ||Du^n|| = \sqrt{((\alpha^n)^2 + (\beta^n)^2)}$

Numerical Results

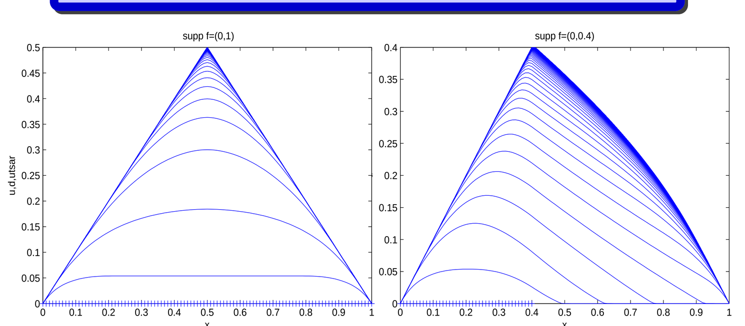


FIGURE 2: The pile support expands before its slope becomes critical,as expected from the physical behaviour,f=.5.N=100

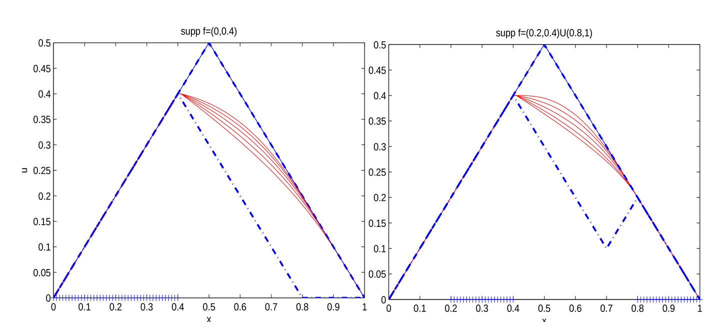


FIGURE 3: f=.5/1/1.5/2,plots of stationary $u_i[-]$ compared with distance function $d[-]$ and the minimal solution $u^*[-.-]$

$$0 \le f_1 \le f_2 \text{ in supp} f \Rightarrow u^* \le u_1 \le u_2 \le d \text{ in}[0,1]$$

Comparsion of Steady states and CFL with [FV06]

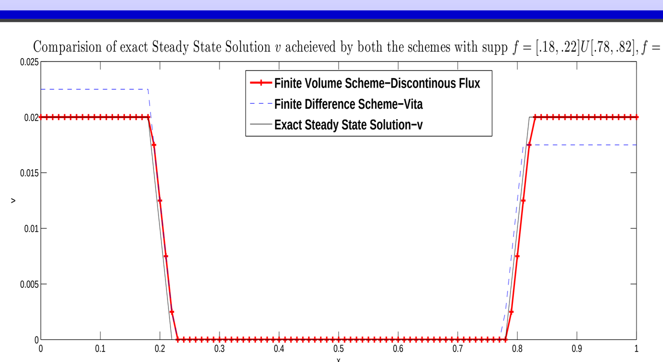


FIGURE 4: Comparision of exact steady state v with steady state v by (3) and (4) and finite difference scheme by [FV06]

Discontinuous Flux Scheme

N	f=1	f=2	f=3
201	.995	.995	.67
401	.9975	.9975	.6683
601	.9983	.9983	.6678
801	.9988	.9988	.6675

Finte Difference Scheme[FV06]

N	f=1	f=2	f=3
201	.6	.52	.46
401	.552	.476	.424
601	.528	.45	.41
801	.5	.43	.39

- Since,$\max_{[0,1]} v$ depends on f,the CFL decreases as f goes higher,but our scheme doesn't make it decrease as rapidly as mentioned in [FV06].CFL $\lambda \le \min(.5, \frac{C}{||f||_\infty})$ where$||v||_\infty \le C||f||_\infty$ is used.
- HOW TO DETERMINE C AT EACH TIME STEP?
- [FV06] uses $C = 6$.It does not work for all values of supp$f$ and for large values of f.

Numerical Experiments: Dimension 2

Discontinuous Flux Scheme

N	f=.5	$f = max(0, sin(2\pi x) + cos(2\pi y) - sin(2\pi xy)$
51	.9	.9
101	.8979	.7354
201	.8058	.6448

Finte Difference Scheme[FV06]

N	f=.5	$f = max(0, sin(2\pi x) + cos(2\pi y) - sin(2\pi xy)$
51	.7	.6
101	.6	.5
201	.4	.4

FIGURE 5: Comparision of CFL needed to acheive exact steady state v by(3) and (4) and finite difference scheme by [FV06] in 2 dimension in case support$f = .5$ on $[0,1] \times [0.1]$

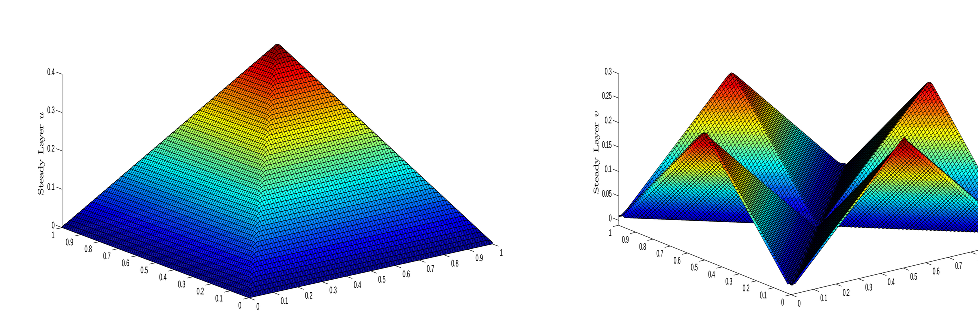


FIGURE 6: Sanpile for $f = .5$

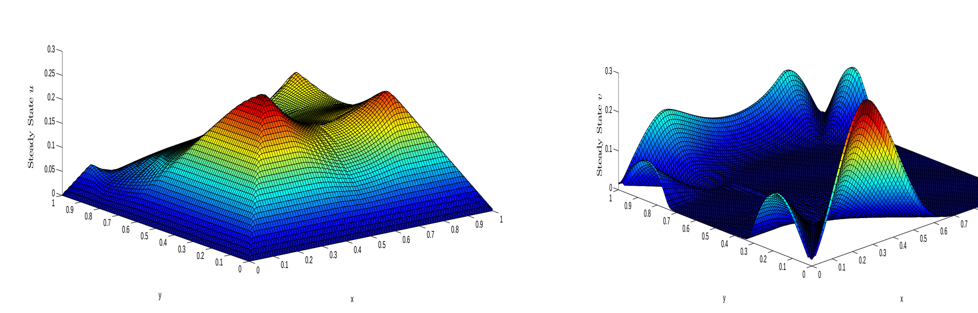


FIGURE 7: Sanpile for $f = \max(0, sin(2\pi x) + cos(2\pi y) - sin(2\pi xy)$

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